

CONTRAVARIANCE THROUGH ENRICHMENT

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ABSTRACT. We define strict and weak duality involutions on 2-categories, and prove a coherence theorem that every bicategory with a weak duality involution is biequivalent to a 2-category with a strict duality involution. For this purpose we introduce “2-categories with contravariance”, a sort of enhanced 2-category with a basic notion of “contravariant morphism”, which can be regarded either as generalized multicategories or as enriched categories. This enables a universal characterization of duality involutions using absolute weighted colimits, leading to a conceptual proof of the coherence theorem.

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1. INTRODUCTION

One of the more mysterious bits of structure possessed by the 2-category \mathcal{Cat} is its *duality involution*

$$(-)^{\text{op}} : \mathcal{Cat}^{\text{co}} \rightarrow \mathcal{Cat}.$$

(As usual, the notation $(-)^{\text{co}}$ denotes reversal of 2-cells but not 1-cells.) Many familiar 2-categories possess similar involutions, such as 2-categories of enriched or internal categories, the 2-category of monoidal categories and strong monoidal functors, or $[\underline{A}, \mathcal{Cat}]$ whenever \underline{A} is a locally groupoidal 2-category; and they are an essential part of much standard category theory.

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However, there does not yet exist a complete abstract theory of such “duality involutions”. A big step forward was the observation by Day and Street [DS97] that A^{op} is a monoidal dual of A in the monoidal bicategory of profunctors. As important and useful as this fact is, it does not exhaust the properties of $(-)^{\text{op}}$; indeed, it does not even determine A^{op} up to equivalence!

In this paper we study duality involutions like $(-)^{\text{op}}$ acting on 2-categories like \mathcal{Cat} , rather than bicategories like \mathcal{Prof} . (We leave it for future work to combine the two, perhaps with a theory of “duality involutions on proarrow equipments”.) Note that in most of the examples cited above, $(-)^{\text{op}}$ is a 2-functor that is a *strict* involution, in that we have $(A^{\text{op}})^{\text{op}} = A$ on the nose. On the other hand, from a higher-categorical perspective it would be more natural to ask only for a *weak* duality involution, where $(-)^{\text{op}}$ is a pseudofunctor that is self-inverse up to coherent pseudo-natural equivalence. For instance, strict duality involutions are not preserved by passage to a biequivalent bicategory, but weak ones are.

The main result of this paper is that there is no loss of generality in considering only strict involutions. More precisely, we prove the following coherence theorem.

Theorem 1.1. *Every bicategory with a weak duality involution is biequivalent to a 2-category with a strict duality involution, by a biequivalence which respects the involutions up to coherent equivalence.*

Let me now say a few words about the proof of Theorem 1.1, which I regard as more interesting than its statement. Often, when proving a coherence theorem for categorical structure at the level of *objects*, it is helpful to consider first an additional structure at the level of *morphisms*, whose presence enables the object-level structure to be characterized by a universal property. For instance, instead of pseudofunctors $A^{\text{op}} \rightarrow \mathcal{Cat}$, we may consider categories over A , among which those underlying some pseudofunctor (the fibrations) are characterized by the existence of cartesian arrows, which have a universal property. Similarly, instead of monoidal categories, we may consider multicategories, among which those underlying some monoidal category are characterized by the existence of representing objects, which also have a universal property.

An abstract framework for this procedure is the theory of *generalized multicategories*; see [Her01, CS10] and the numerous other references in [CS10]. In general, for a suitably nice 2-monad T , in addition to the usual notions of strict and pseudo T -algebra, there is a notion of *virtual T -algebra*, which contains additional kinds of morphisms whose domain “ought to be an object given by a T -action if such existed”. For example, if T is the 2-monad for strict monoidal categories, then a virtual T -algebra is an ordinary multicategory, in which there are “multimorphisms” whose domains are finite lists of objects that “ought to be tensor products if we had a monoidal category”.

In our case, it is easy to write down a 2-monad whose strict algebras are 2-categories with a strict duality involution: it is $T\mathcal{A} = \mathcal{A} + \mathcal{A}^{\text{co}}$. A virtual algebra for this 2-monad is, roughly speaking, a 2-category equipped with a basic notion of “contravariant morphism”. That is, for each pair of objects x and y , there are two hom-categories $\underline{\mathcal{A}}^+(x, y)$ and $\underline{\mathcal{A}}^-(x, y)$, whose objects we call *covariant* and *contravariant* morphisms respectively. Composition is defined in the obvious way: the composite of two morphisms of the same variance is covariant, while the composite of two morphisms of different variances is contravariant. In addition,

postcomposing with a contravariant morphism is contravariant on 2-cells. We call such a gadget a **2-category with contravariance**.

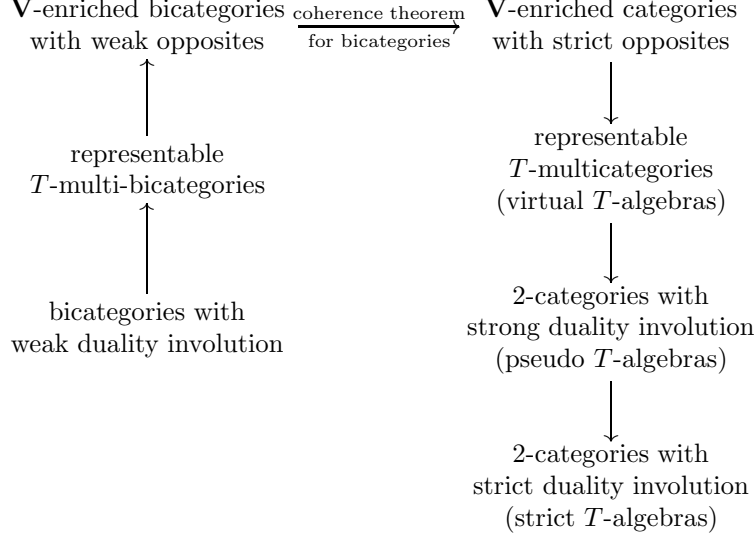
Now, as with any sort of generalized multicategory, we can characterize the virtual T -algebras that are pseudo T -algebras by a notion of *representability*. This means that for each object x , we have an object x° and isomorphisms $\underline{\mathbf{A}}^-(x, y) \cong \underline{\mathbf{A}}^+(x^\circ, y)$ and $\underline{\mathbf{A}}^+(x, y) \cong \underline{\mathbf{A}}^-(x^\circ, y)$, jointly natural in y . We call an object x° with this property a (*strict*) *opposite* of x . The corresponding pseudo T -algebra structure describes this operation $(-)^\circ$ as a *strong* duality involution on the underlying 2-category $\underline{\mathbf{A}}^+$, meaning a strict 2-functor $(\underline{\mathbf{A}}^+)^{\text{co}} \rightarrow \underline{\mathbf{A}}^+$ that is self-inverse up to coherent strict 2-natural isomorphism.

Now, it turns out that 2-categories with contravariance are not just generalized multicategories: they are also *enriched categories*. Namely, there is a (non-symmetric) monoidal category, denoted \mathbf{V} (for Variance), such that \mathbf{V} -enriched categories are the same as 2-categories with contravariance. (As a category, \mathbf{V} is just $\mathbf{Cat} \times \mathbf{Cat}$, but its monoidal structure is not the usual one.) From this perspective, we can alternatively describe strict opposites as *weighted colimits*: x° is the copower (or “tensor”) of x by a particular object $\mathbb{1}^-$ of \mathbf{V} , called the *dual unit*. Since $\mathbb{1}^-$ is dualizable in \mathbf{V} , opposites are an *absolute* or *Cauchy* colimit in the sense of [Str83]: they are preserved by all \mathbf{V} -enriched functors. It follows that any 2-category-with-contravariance has a “completion” with respect to opposites, and this operation is idempotent.

We have now moved into a context having a straightforward bicategorical version. We simply observe that \mathbf{V} can be made into a monoidal 2-category, and consider \mathbf{V} -enriched bicategories; we call these *bicategories with contravariance*. In such a bicategory we can consider “weak opposites”, asking only for pseudonatural equivalences $\underline{\mathbf{A}}^-(x, y) \simeq \underline{\mathbf{A}}^+(x^\circ, y)$ and $\underline{\mathbf{A}}^+(x, y) \simeq \underline{\mathbf{A}}^-(x^\circ, y)$; these are “absolute weighted bicolimits” in the sense of [GS16]. Since any isomorphism of categories is an equivalence, any strict opposite is also a weak one. (More abstractly, strict opposites should be *flexible colimits* [BKPS89] in a suitable sense, but we will not make this precise.)

Now, it is straightforward to generalize the coherence theorem for bicategories to a coherence theorem for *enriched* bicategories. Therefore, any bicategory with contravariance is biequivalent to a 2-category with contravariance. This suggests that the process by which we arrived at \mathbf{V} -enriched categories could be duplicated on the bicategorical side, yielding the following “ladder” strategy for proving

Theorem 1.1:



There are three problems with this idea, two minor and one major. The first is that it (apparently) produces only a strong duality involution rather than a strict one, necessitating an extra step at the bottom-right of the ladder, as shown. However, the strictification of pseudo-algebras for 2-monads is fairly well-understood, so we can apply a general coherence result [Pow89, Lac02].

The second problem is that *a priori*, the coherence theorem for \mathbf{V} -enriched bicategories does not also strictify the weak opposites into strict opposites. However, this is also easy to remedy: since the strictification of a \mathbf{V} -bicategory with weak opposites will still have weak opposites, and any strict opposite is also a weak one, it will be biequivalent to its free cocompletion under strict opposites.

The third, and more major, problem with this strategy is that there is no extant theory of “generalized multi-bicategories”. We could develop such a theory, but it would take us rather far afield. Thus, instead we will “hop over” that rung of the ladder by constructing a \mathbf{V} -enriched bicategory with weak opposites directly from a bicategory with a weak duality involution, by a “beta-reduced” and weakened version of the analogous operation on the other side.

Since this direct construction also includes the strict case, we could, formally speaking, dispense with the multicategories on the other side as well. Indeed, the entire proof can be beta-reduced into a more compact form: if we prove the coherence theorem for enriched bicategories using a Yoneda embedding, the strictification and cocompletion processes could be combined into one and tweaked slightly to give a strict duality involution directly.

In fact, there are not many applications of [Theorem 1.1](#) anyway. First of all, it is not all that easy to think of naturally occurring duality involutions that are not already strict. But here are a few:

- (1) The 2-category of fibrations over some base category \mathbf{S} has a “fiberwise” duality involution, but since its action on non-vertical arrows has to be constructed in a more complicated way than simply turning them around, it is not strict.

- (2) If \mathcal{B} is a compact closed bicategory [DS97, Sta13], then its bicategory $\mathcal{M}ap(\mathcal{B})$ of maps (left adjoints) has a duality involution that is not generally strict.
- (3) If \mathcal{A} is a bicategory with a duality involution, and \mathcal{W} is a class of morphisms in \mathcal{A} admitting a calculus of fractions [Pro96] and closed under the duality involution, then the bicategory of fractions $\mathcal{A}[\mathcal{W}^{-1}]$ inherits a duality involution that is not strict (even if the one on \mathcal{A} was strict).

However, even in these cases [Theorem 1.1](#) is not as important as it might be, because Lack’s coherence theorem (“naturally occurring bicategories are biequivalent to naturally occurring 2-categories”) applies very strongly to duality involutions: nearly all naturally occurring bicategories with duality involutions are biequivalent to some *naturally occurring* strict 2-category with a strict duality involution. For the examples above, we have:

- (1) The 2-category of fibrations over \mathbf{S} is biequivalent to the 2-category of \mathbf{S} -indexed categories, which has a strict duality involution inherited from \mathbf{Cat} .)
- (2) For the standard examples of compact closed bicategories such as $\mathcal{P}rof$ or $\mathcal{S}pan$, the bicategory of maps is biequivalent to a well-known strict 2-category with a strict duality involution, such as $\mathcal{C}at_{cc}$ or \mathbf{Set} .
- (3) Many naturally occurring examples of bicategories of fractions are also biequivalent to well-known 2-categories with strict duality involutions, such as some 2-category of stacks.

Thus, if [Theorem 1.1](#) were the main point of this paper, it would be somewhat disappointing. However, I regard the method of proof, and the entire ladder it gives rise to, as more important than the result itself. Representating contravariance using generalized multicategories and enrichment seems a promising avenue for future study of further properties of duality involutions. From this perspective, the paper is primarily a contribution to *enhanced 2-category theory* in the sense of [LS12], which just happens to prove a coherence theorem to illustrate the ideas.

Furthermore, our abstract approach also generalizes to other types of contravariance. The right-hand side of the ladder, at least, works in the generality of any group action on any monoidal category \mathbf{W} . The motivating case of duality involutions on 2-categories is the case when $\mathbb{Z}/2\mathbb{Z}$ acts on \mathbf{Cat} by $(-)^{op}$; but other actions representing other kinds of contravariance include:

- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbf{2Cat}$ by $(-)^{op}$ and $(-)^{co}$. When $\mathbf{2Cat}$ is given the Gray monoidal structure, this yields a theory of duality involutions on Gray-categories.
- $(\mathbb{Z}/2\mathbb{Z})^n$ acts on strict n -categories (including the case $n = \omega$), yielding duality involutions for strict $(n + 1)$ -categories.
- $\mathbb{Z}/2\mathbb{Z}$ acts on the category \mathbf{sSet} of simplicial sets by reversing the directions of all the simplices. With simplicial sets modeling $(\infty, 1)$ -categories as quasi-categories, this yields a theory of duality involutions on a particular model for $(\infty, 2)$ -categories (see for instance [RV15]).
- Combining the ideas of the last two examples, $(\mathbb{Z}/2\mathbb{Z})^n$ acts on the category of Θ_n -spaces by reversing direction at all dimensions, leading to duality involutions on an enriched-category model for $(\infty, n + 1)$ -categories [BR13].

We will not develop any of these examples further here, but the perspective of describing contravariance through enrichment may be useful for all of them as well.

We begin in [section 2](#) by defining weak, strong, and strict duality involutions. Then we proceed up the ladder from the bottom right. In [section 3](#) we express strong and strict duality involutions as algebra structures for a 2-monad, and deduce that strong ones can be strictified. In [section 4](#) we express strong duality involutions using generalized multicategories, and in [sections 5–6](#) we reexpress them using enrichment. In [section 7](#) we jump over to the other side of the ladder, showing that weak duality involutions on bicategories can be expressed using bicategorical enrichment. Then finally in [section 8](#) we cross the top of the ladder with a coherence theorem for enriched bicategories.

2. DUALITY INVOLUTIONS

In this section define strict, strong, and weak duality involutions, allowing us to state [Theorem 1.1](#) precisely.

Definition 2.1. A **weak duality involution** on a bicategory \mathcal{A} consists of:

- A pseudofunctor $(-)^{\circ} : \mathcal{A}^{\text{co}} \rightarrow \mathcal{A}$.
- A pseudonatural adjoint equivalence

$$\begin{array}{ccc} \mathcal{A} & \xlongequal{\quad} & \mathcal{A} \\ & \Downarrow \eta & \\ ((-)^{\circ})^{\text{co}} & \xrightarrow{\quad} & (-)^{\circ} \end{array}$$

- An invertible modification

$$\begin{array}{ccc} \mathcal{A}^{\text{co}} \xrightarrow{(-)^{\circ}} \mathcal{A} & \xlongequal{\quad} & \mathcal{A} \\ & \Downarrow \eta & \\ ((-)^{\circ})^{\text{co}} & \xrightarrow{\quad} & (-)^{\circ} \end{array} \xRightarrow{\zeta} \begin{array}{ccc} \mathcal{A}^{\text{co}} & \xlongequal{\quad} & \mathcal{A}^{\text{co}} \xrightarrow{(-)^{\circ}} \mathcal{A} \\ & \Downarrow \eta^{\text{co}} & \\ (-)^{\circ} & \xrightarrow{\quad} & ((-)^{\circ})^{\text{co}} \end{array}$$

whose components are therefore 2-cells $\zeta_x : \eta_{x^{\circ}} \xrightarrow{\sim} (\eta_x)^{\circ}$.

- For any $x \in \mathcal{A}$, we have

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & x^{\circ\circ} \\ & \searrow \eta_{x^{\circ\circ}} & \nearrow (\eta_x)^{\circ} \\ & x^{\circ\circ\circ} & \end{array} \quad = \quad \begin{array}{ccc} & x^{\circ\circ} & \\ \eta_x \nearrow & \Downarrow \cong & \searrow \eta_{x^{\circ\circ}} \\ x & \xrightarrow{\eta_x} & x^{\circ\circ} \\ & \searrow \eta_{x^{\circ\circ}} & \nearrow (\eta_x)^{\circ} \\ & x^{\circ\circ\circ} & \end{array}$$

(the unnamed isomorphism is a pseudonaturality constraint for η).

If \mathcal{A} is a strict 2-category, a **strong duality involution** on \mathcal{A} is a weak duality involution for which

- $(-)^{\circ}$ is a strict 2-functor,
- η is a strict 2-natural isomorphism, and
- ζ is an identity.

If moreover η is an identity, we call it a **strict duality involution**.

In particular, η and ζ in a weak duality involution exhibit $(-)^{\circ}$ and $((-)^{\circ})^{\text{co}}$ as a biadjoint biequivalence between \mathcal{A} and \mathcal{A}^{co} , in the sense of [Gur12]. Similarly, in a strong duality involution, η exhibits $(-)^{\circ}$ and $((-)^{\circ})^{\text{co}}$ as a 2-adjoint 2-equivalence between \mathcal{A} and \mathcal{A}^{co} . And, of course, in a strict duality involution, $(-)^{\circ}$ and $((-)^{\circ})^{\text{co}}$ are inverse isomorphisms of 2-categories.

Definition 2.2. If \mathcal{A} and \mathcal{B} are bicategories equipped with weak duality involutions, a **duality pseudofunctor** $F : \mathcal{A} \rightarrow \mathcal{B}$ is a pseudofunctor equipped with

- A pseudonatural adjoint equivalence

$$\begin{array}{ccc} \mathcal{A}^{\text{co}} & \xrightarrow{F^{\text{co}}} & \mathcal{B}^{\text{co}} \\ (-)^{\circ} \downarrow & \Downarrow_{\mathbf{i}} & \downarrow (-)^{\circ} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

- An invertible modification

$$\begin{array}{ccc} & \mathcal{A} & \xrightarrow{F} \mathcal{B} \\ & \nwarrow^{((-)^{\circ})^{\text{co}}} & \swarrow_{((-)^{\circ})^{\text{co}}} \\ \mathcal{A}^{\text{co}} & \xrightarrow{F^{\text{co}}} & \mathcal{B}^{\text{co}} \\ (-)^{\circ} \downarrow & \Downarrow_{\mathbf{i}} & \downarrow (-)^{\circ} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \quad \xRightarrow{\theta} \quad \begin{array}{ccc} & \mathcal{A} & \\ & \nwarrow^{((-)^{\circ})^{\text{co}}} & \\ \mathcal{A}^{\text{co}} & & \\ (-)^{\circ} \downarrow & \Downarrow_{\eta} & \\ \mathcal{A} & & \end{array}$$

whose components are therefore 2-cells in \mathcal{B} of the following shape:

$$\begin{array}{ccc} (Fx)^{\circ\circ} & \xrightarrow{(i_x)^{\circ}} & (F(x^{\circ}))^{\circ} \\ \eta_{Fx} \uparrow & \Downarrow_{\theta_x} & \downarrow i_x^{\circ} \\ Fx & \xrightarrow{F(\eta_x)} & F(x^{\circ\circ}) \end{array}$$

- For any $x \in \mathcal{A}$, we have

$$\begin{array}{ccc} (Fx)^{\circ\circ\circ} & \xrightarrow{(i_x)^{\circ\circ}} & (F(x^{\circ}))^{\circ\circ} \\ \eta_{(Fx)^{\circ}} \left(\begin{array}{c} \xrightarrow{\zeta_{Fx}} \\ \xleftarrow{\zeta_{Fx}} \end{array} \right) & & \downarrow (i_x)^{\circ} \\ (Fx)^{\circ} & \xrightarrow{(F\eta_x)^{\circ}} & (F(x^{\circ\circ}))^{\circ} \\ \downarrow i_x & \Downarrow_{\cong} & \downarrow i_x^{\circ\circ} \\ F(x^{\circ}) & \xrightarrow{F((\eta_x)^{\circ})} & F(x^{\circ\circ\circ}) \end{array} = \begin{array}{ccccc} (Fx)^{\circ\circ\circ} & \xrightarrow{(i_x)^{\circ\circ}} & (F(x^{\circ}))^{\circ\circ} & \xrightarrow{(i_x^{\circ})^{\circ}} & (F(x^{\circ\circ}))^{\circ} \\ \uparrow \eta_{(Fx)^{\circ}} & \Downarrow_{\cong} & \uparrow \eta_{F(x^{\circ})} & \Downarrow_{\theta_{x^{\circ}}} & \downarrow i_{x^{\circ\circ}} \\ (Fx)^{\circ} & \xrightarrow{i_x} & F(x^{\circ}) & \xrightarrow{F(\eta_x)^{\circ}} & F(x^{\circ\circ\circ}) \\ & & \downarrow i_x^{\circ} & \Downarrow_{F(\zeta_x)} & \\ & & F(x^{\circ}) & \xrightarrow{F((\eta_x)^{\circ})} & F(x^{\circ\circ\circ}) \end{array}$$

(the unnamed isomorphisms are pseudonaturality constraints for \mathbf{i} and η).

If \mathcal{A} and \mathcal{B} are strict 2-categories with strong duality involutions, then a **(strong) duality 2-functor** $F : \mathcal{A} \rightarrow \mathcal{B}$ is a duality pseudofunctor such that

- F is a strict 2-functor,
- \mathbf{i} is a strict 2-natural isomorphism, and
- θ is an identity.

If \mathbf{i} is also an identity, we call it a **strict duality 2-functor**.

Note that if the duality involutions of \mathcal{A} and \mathcal{B} are strict, then the identity θ says that $(\mathbf{i}_x)^\circ = (\mathbf{i}_x)'^{-1}$. On the other hand, if \mathcal{A} is a strict 2-category with two strong duality involutions $(-)^\circ$ and $(-)'^\circ$, to make the identity 2-functor into a duality 2-functor is to give a natural isomorphism $A^\circ \cong A'^\circ$ that commutes with the isomorphisms η and η' .

Now [Theorem 1.1](#) can be stated more precisely as:

Theorem 2.3. *If \mathcal{A} is a bicategory with a weak duality involution, then there is a 2-category \mathcal{A}' with a strict duality involution and a duality pseudofunctor $\mathcal{A} \rightarrow \mathcal{A}'$ that is a biequivalence.*

We could make this more algebraic by defining a whole tricategory of bicategories with weak duality involution and showing that our biequivalence lifts to an internal biequivalence therein, but we leave that to the interested reader. In fact, the correct definitions of transformations and modifications can be extracted from our characterization via enrichment. (It turns out to only be possible to define invertible modifications.)

Remark 2.4. The definition of duality involution may seem a little *ad hoc*. In [section 7](#) we will rephrase it as a special case of a “twisted group action”, which may make it seem more natural.

We end this section with some examples.

Example 2.5. With nearly any reasonable set-theoretic definition of “category” and “opposite”, \mathbf{Cat} has a strict duality involution. The same is true for the 2-category of categories enriched over any symmetric monoidal category, or the 2-category of categories internal to some category with pullbacks.

Example 2.6. If \mathcal{A} is a bicategory with a weak duality involution and \mathcal{K} is a locally groupoidal bicategory, then the bicategory $[\mathcal{K}, \mathcal{A}]$ of pseudofunctors, pseudonatural transformations, and modifications inherits a weak duality involution by applying the duality involution of \mathcal{A} pointwise. Local groupoidality of \mathcal{K} ensures that $\mathcal{K} \cong \mathcal{K}^{\text{co}}$, so that we can define the dual of a pseudofunctor $F : \mathcal{K} \rightarrow \mathcal{A}$ to be

$$F^\circ : \mathcal{K} \cong \mathcal{K}^{\text{co}} \xrightarrow{F^{\text{co}}} \mathcal{A}^{\text{co}} \xrightarrow{(-)^\circ} \mathcal{A}.$$

The rest of the structure follows by whiskering.

Example 2.7. If \mathcal{A} is a bicategory with a weak duality involution and $F : \mathcal{A} \rightarrow \mathcal{C}$ is a biequivalence, then \mathcal{C} can be given a weak duality involution making F a duality pseudofunctor. We first have to enhance F to a biadjoint biequivalence as in [\[Gur12\]](#); then we define all the structure by composing with F and its inverse.

Example 2.8. The 2-category of fibrations over a base category \mathbf{S} has a strong duality involution constructed as follows. Given a fibration $P : \mathbf{C} \rightarrow \mathbf{S}$, in its dual $P^\circ : \mathbf{C}^\circ \rightarrow \mathbf{S}$ the objects of \mathbf{C}° are those of \mathbf{C} , while the morphisms from x to y

over a morphism $f : a \rightarrow b$ in \mathbf{S} are the morphisms $f^*y \rightarrow x$ over a in \mathbf{C} . Here f^*y denotes the pullback of y along f obtained from some cartesian lifting; the resulting “set of morphisms from x to y ” in \mathbf{C}° is independent, up to canonical isomorphism, of the choice of cartesian lift. However, there is no obvious way to define it such that $\mathbf{C}^{\circ\circ}$ is *equal* to \mathbf{C} , rather than merely canonically isomorphic. Of course, the 2-category of fibrations over \mathbf{S} is biequivalent to the 2-category of \mathbf{S} -indexed categories, which has a strict duality involution induced from its codomain \mathbf{Cat} .

Example 2.9. Let \mathcal{A} be a bicategory with a duality involution, let \mathcal{W} be a class of morphisms of \mathcal{A} admitting a *calculus of right fractions* in the sense of [Pro96], and suppose moreover that if $v \in \mathcal{W}$ then $v^\circ \in \mathcal{W}$. Then the bicategory of fractions $\mathcal{A}[\mathcal{W}^{-1}]$ also admits a duality involution, constructed using its universal property [Pro96, Theorem 21] as follows.

Let $\ell : \mathcal{A} \rightarrow \mathcal{A}[\mathcal{W}^{-1}]$ be the localization functor. By assumption, the composite $\mathcal{A} \xrightarrow{((-)^\circ)^{\text{co}}} \mathcal{A}^{\text{co}} \xrightarrow{\ell^{\text{co}}} \mathcal{A}[\mathcal{W}^{-1}]^{\text{co}}$ takes morphisms in \mathcal{W} to equivalences. Thus, it factors through ℓ , up to equivalence, by a functor that we denote $((-)^\diamond)^{\text{co}} : \mathcal{A}[\mathcal{W}^{-1}] \rightarrow \mathcal{A}[\mathcal{W}^{-1}]^{\text{co}}$ (that is, a functor whose 2-cell dual we denote $(-)^\diamond$). Now the pasting composite composite

is a pseudonatural equivalence from ℓ to $(-)^\diamond \circ ((-)^\diamond)^{\text{co}} \circ \ell$. Hence, by the universal property of ℓ , it is isomorphic to the whiskering by ℓ of some pseudonatural equivalence

Similar whiskering arguments produce the modification ζ' and verify its axiom. Note that this induced duality involution on $\mathcal{A}[\mathcal{W}^{-1}]$ will not generally be strict, even if the one on \mathcal{A} is. (On the other hand, as remarked in section 1, often $\mathcal{A}[\mathcal{W}^{-1}]$ is biequivalent to some naturally-occurring 2-category having a strict duality involution, such as the examples of étendues and stacks considered in [Pro96].)

Example 2.10. Let \mathcal{B} be a *compact closed bicategory* (also called *symmetric autonomous*) as in [DS97, Sta13]. Thus means that \mathcal{B} is symmetric monoidal, and moreover each object x has a dual x° with respect to the monoidal structure, with morphisms $\eta : \mathbb{1} \rightarrow x \otimes x^\circ$ and $\varepsilon : x^\circ \otimes x \rightarrow \mathbb{1}$ satisfying the triangle identities up to isomorphism. If we choose such a dual for each object, then $(-)^\circ$ can be made into a biequivalence $\mathcal{B}^{\text{op}} \rightarrow \mathcal{B}$, sending a morphism $g : y \rightarrow x$ to the composite

$$x^\circ \xrightarrow{\eta_y} x^\circ \otimes y \otimes y^\circ \xrightarrow{g} x^\circ \otimes x \otimes y^\circ \xrightarrow{\varepsilon_x} y^\circ,$$

with η and ε becoming pseudonatural transformations. Moreover, this functor $\mathcal{B}^{\text{op}} \rightarrow \mathcal{B}$ looks exactly like a duality involution except that $(-)^{\text{co}}$ has been replaced by $(-)^{\text{op}}$: we have a pseudonatural adjoint equivalence

$$\begin{array}{ccc} \mathcal{B} & \xlongequal{\quad} & \mathcal{B} \\ \downarrow \eta & & \downarrow \eta \\ ((-)^{\text{op}})^{\text{op}} & \xrightarrow{\quad} & \mathcal{B}^{\text{op}} \xrightarrow{(-)^{\text{op}}} \mathcal{B} \end{array}$$

and an invertible modification

$$\begin{array}{ccc} \mathcal{B}^{\text{op}} \xrightarrow{(-)^{\text{op}}} \mathcal{B} & \xlongequal{\quad} & \mathcal{B} \\ \downarrow \eta & & \downarrow \eta \\ ((-)^{\text{op}})^{\text{op}} & \xrightarrow{\quad} & \mathcal{B}^{\text{op}} \xrightarrow{(-)^{\text{op}}} \mathcal{B} \end{array} \xRightarrow{\zeta} \begin{array}{ccc} \mathcal{B}^{\text{op}} & \xlongequal{\quad} & \mathcal{B}^{\text{op}} \xrightarrow{(-)^{\text{op}}} \mathcal{B} \\ \downarrow \eta^{\text{op}} & & \downarrow \eta^{\text{op}} \\ (-)^{\text{op}} & \xrightarrow{\quad} & \mathcal{B} \end{array}$$

satisfying the same axiom as in [Definition 2.1](#). Explicitly, η is the composite

$$x \xrightarrow{x \otimes \eta_x^{\text{op}}} x \otimes x^{\text{op}} \otimes x^{\text{op}} \xrightarrow{\sim} x^{\text{op}} \otimes x \otimes x^{\text{op}} \xrightarrow{\varepsilon_x \otimes x^{\text{op}}} x^{\text{op}}$$

and ζ is obtained as a pasting composite

$$\begin{array}{ccccc} & x^{\text{op}} x^{\text{op}} x^{\text{op}} & \xrightarrow{\sim} & x^{\text{op}} x^{\text{op}} x^{\text{op}} & \\ x^{\text{op}} \nearrow x^{\text{op}} \eta_{x^{\text{op}}} & \downarrow \cong & & \downarrow \cong & \searrow \varepsilon_{x^{\text{op}}} x^{\text{op}} \\ x^{\text{op}} & & & & x^{\text{op}} \\ & \downarrow (x \eta_{x^{\text{op}}})^{\text{op}} & & \downarrow (\varepsilon_x x^{\text{op}})^{\text{op}} & \\ & (xx^{\text{op}} x^{\text{op}})^{\text{op}} & \xrightarrow{\sim} & (x^{\text{op}} x x^{\text{op}})^{\text{op}} & \end{array}$$

using the fact that if x^{op} is a dual of x , then by symmetry of \mathcal{B} , x is a dual of x^{op} .

Now let \mathcal{A} be the locally full sub-bicategory of maps (left adjoints) in \mathcal{B} . Since passing from left to right adjoints reverses the direction of 2-cells, we have a “take the right adjoint” functor $\mathcal{A}^{\text{coop}} \rightarrow \mathcal{B}$, or equivalently $\mathcal{A}^{\text{co}} \rightarrow \mathcal{B}^{\text{op}}$. Composing with the above “duality” functor $\mathcal{B}^{\text{op}} \rightarrow \mathcal{B}$, we have a functor $\mathcal{A}^{\text{co}} \rightarrow \mathcal{B}$, and since right adjoints in \mathcal{B} are left adjoints in \mathcal{B}^{op} , this functor lands in \mathcal{A} , giving $(-)^{\text{op}} : \mathcal{A}^{\text{co}} \rightarrow \mathcal{A}$. Of course, equivalences are maps, so the above η and ζ lie in \mathcal{A} , and therefore equip \mathcal{A} with a duality involution.

This duality involution on \mathcal{A} is not generally strict or even strong. However, as remarked in [section 1](#), in many naturally-occurring examples \mathcal{A} is equivalent to some naturally-occurring 2-category with a strict duality involution. For instance, if $\mathcal{B} = \mathcal{P}rof$ then $\mathcal{A} \simeq \mathcal{C}at_{\text{cc}}$, the 2-category of Cauchy-complete categories; while if $\mathcal{B} = \mathcal{S}pan$ then $\mathcal{A} \simeq \mathcal{S}et$, and similarly for internal and enriched versions.

3. A 2-MONADIC APPROACH

Let $T(\mathcal{A}) = \mathcal{A} + \mathcal{A}^{\text{co}}$, an endo-2-functor of the 2-category $2\text{-}\mathcal{C}at$ of 2-categories, strict 2-functors, and strict 2-natural transformations. We have an obvious strict 2-natural transformation $\eta : \text{Id} \rightarrow T$, and we define $\mu : TT \rightarrow T$ by

$$(\mathcal{A} + \mathcal{A}^{\text{co}}) + (\mathcal{A} + \mathcal{A}^{\text{co}})^{\text{co}} \xrightarrow{\sim} \mathcal{A} + \mathcal{A}^{\text{co}} + \mathcal{A}^{\text{co}} + \mathcal{A} \xrightarrow{\nabla} \mathcal{A} + \mathcal{A}^{\text{co}}$$

where ∇ is the obvious “fold” map.

Theorem 3.1. *T is a strict 2-monad, and:*

- (i) Normal pseudo T -algebras are 2-categories with strong duality involutions;
- (ii) Pseudo T -morphisms are duality 2-functors; and
- (iii) Strict T -algebras are 2-categories with strict duality involutions.

Proof. The 2-monad laws for T are straightforward to check. By a normal pseudo algebra we mean one for which the unit constraint identifying $\mathcal{A} \rightarrow T\mathcal{A} \rightarrow \mathcal{A}$ with the identity map is itself an identity. Thus, when $T\mathcal{A} = \mathcal{A} + \mathcal{A}^{\text{co}}$, this means the action $a : T\mathcal{A} \rightarrow \mathcal{A}$ contains no data beyond a 2-functor $(-)^{\circ} : \mathcal{A}^{\text{co}} \rightarrow \mathcal{A}$. The remaining data is a 2-natural isomorphism

$$\begin{array}{ccc} TT\mathcal{A} & \xrightarrow{Ta} & T\mathcal{A} \\ \mu \downarrow & \not\cong & \downarrow a \\ T\mathcal{A} & \xrightarrow{a} & \mathcal{A} \end{array} \quad \text{that is} \quad \begin{array}{ccc} \mathcal{A} + \mathcal{A}^{\text{co}} + \mathcal{A}^{\text{co}} + \mathcal{A} & \longrightarrow & \mathcal{A} + \mathcal{A}^{\text{co}} \\ \downarrow & \not\cong & \downarrow \\ \mathcal{A} + \mathcal{A}^{\text{co}} & \longrightarrow & \mathcal{A} \end{array}$$

satisfying three axioms that can be found, for instance, in [Lac02, §1]. The right-hand square commutes strictly on the first three summands in its domain, and the second and third of the coherence axioms say exactly that the given isomorphism in these cases is an identity. Thus, what remains is the component of the isomorphism on the fourth summand, which has precisely the form of η in Definition 2.1, and it is easy to check that the first coherence axiom reduces in this case to the identity ζ . This proves (i), and (iii) follows immediately.

Similarly, for (ii), a pseudo T -morphism is a 2-functor $F : \mathcal{A} \rightarrow \mathcal{B}$ together with a 2-natural isomorphism

$$\begin{array}{ccc} T\mathcal{A} & \xrightarrow{TF} & T\mathcal{B} \\ \downarrow & \not\cong & \downarrow \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \quad \text{that is} \quad \begin{array}{ccc} \mathcal{A} + \mathcal{A}^{\text{co}} & \xrightarrow{F+F^{\text{co}}} & \mathcal{B} + \mathcal{A}^{\text{co}} \\ \downarrow & \not\cong & \downarrow \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

satisfying two coherence axioms also listed in [Lac02, §1]. This square commutes strictly on the first summand of its domain, and the second coherence axiom ensures that the isomorphism is the identity there. So the remaining data is the isomorphism on the second summand, which has precisely the form of \mathbf{i} in Definition 2.2, and the first coherence axiom reduces to the identity θ . \square

In particular, we obtain an automatic definition of a “duality 2-natural transformation”: a T -2-cell between pseudo T -morphisms. This also gives us another source of examples.

Example 3.2. The 2-category $T\text{-Alg}_s$ of strict T -algebras and strict T -morphisms is complete with limits created in 2-Cat , including in particular Eilenberg–Moore objects [Str72]. Thus, for any monad M in this 2-category — which is to say, a 2-monad that is a strict duality 2-functor and whose unit and multiplication are duality 2-natural transformations — the 2-category $M\text{-Alg}_s$ of strict M -algebras and strict M -morphisms is again a strict T -algebra, i.e. has a strict duality involution.

Similarly, by [BKP89] the 2-category $T\text{-Alg}$ of strict T -algebras and pseudo T -morphisms has PIE-limits, including EM-objects. Thus, we can reach the same conclusion even if M is only a strong duality 2-functor. And since 2-Cat is locally presentable and T has a rank, there is another 2-monad T' whose strict algebras are the pseudo T -algebras; thus we can argue similarly in the 2-category $T\text{-PsAlg}$

of pseudo T -algebras and pseudo T -morphisms, so that strong duality involutions also lift to $M\text{-Alg}_s$.

Usually, of course, we are more interested in the 2-category $M\text{-Alg}$ of strict M -algebras and *pseudo* M -morphisms. It might be possible to enhance the above abstract argument to apply to this case using techniques such as [Lac00, Pow07], but it is easy enough to check directly that if M lies in $T\text{-Alg}$ or $T\text{-PsAlg}$, then so does $M\text{-Alg}$. If (A, a) is an M -algebra, then the induced M -algebra structure on A° is the composite

$$M(A^\circ) \xrightarrow{i} (MA)^\circ \xrightarrow{a^\circ} A^\circ$$

and if $(f, \bar{f}) : (A, a) \rightarrow (B, b)$ is a pseudo M -morphism (where $\bar{f} : a \circ Mf \xrightarrow{\sim} f \circ b$), then f° becomes a pseudo M -morphism with the following structure 2-cell:

$$\begin{array}{ccccc} M(B^\circ) & \xrightarrow{i} & (MB)^\circ & \xrightarrow{b^\circ} & B^\circ \\ \uparrow M(f^\circ) & \Downarrow \cong & \uparrow (Mf)^\circ & \Downarrow (\bar{f}^{-1})^\circ & \uparrow f^\circ \\ M(A^\circ) & \xrightarrow{i} & (MA)^\circ & \xrightarrow{a^\circ} & A^\circ \end{array}$$

The axiom θ of i (which is an equality since M is a strong duality 2-functor) ensures that η lifts to $M\text{-Alg}$ (indeed, to $M\text{-Alg}_s$), and its own θ axiom is automatic. A similar argument applies to $M\text{-PsAlg}$. Thus, 2-categories of algebraically structured categories such as monoidal categories, braided or symmetric monoidal categories, and so on, admit strict duality involutions, even when their morphisms are of the pseudo sort. (Of course, this is impossible for lax or colax morphisms, since dualizing the categories involved would switch lax with colax.)

In theory, this could be another source of weak duality involutions that are not strong: if for a 2-monad M the transformation i were not a strictly 2-natural isomorphism or its axiom θ were not strict, then $M\text{-Alg}$ would only inherit a weak duality involution, even if the duality involution on the original 2-category were strict. However, I do not know any examples of 2-monads that behave this way.

We end this section with the strong-to-strict coherence theorem.

Theorem 3.3. *If \mathcal{A} is a 2-category with a strong duality involution, then there is a 2-category \mathcal{A}' with a strict duality involution and a duality 2-functor $\mathcal{A} \rightarrow \mathcal{A}'$ that is a 2-equivalence.*

Proof. The 2-category 2-Cat admits a factorization system $(\mathcal{E}, \mathcal{M})$ in which \mathcal{E} consists of the 2-functors that are bijective on objects and \mathcal{M} of the 2-functors that are 2-fully-faithful, i.e. an isomorphism on hom-categories. Moreover, this factorization system satisfies the assumptions of [Lac02, Theorem 4.10], and we have $T\mathcal{E} \subseteq \mathcal{E}$. Thus, [Lac02, Theorem 4.10] (which is an abstract version of [Pow89, Theorem 3.4]), together with the characterizations of Theorem 3.1, implies the desired result. \square

Inspecting the proof of the general coherence theorem, we obtain a concrete construction of \mathcal{A}' : it is the result of factoring the pseudo-action map $T\mathcal{A} \rightarrow \mathcal{A}$ as a bijective-on-objects 2-functor followed by a 2-fully-faithful one. In other words, the objects of \mathcal{A}' are two copies of the objects of \mathcal{A} , one copy representing each object and one its opposite, with the duality interchanging them. The morphisms and 2-cells are then easy to determine.

It remains, therefore, to pass from a weak duality involution on a bicategory to a strong one on a 2-category. We proceed up the right-hand side of the ladder from [section 1](#).

4. CONTRAVARIANCE THROUGH VIRTUALIZATION

As mentioned in [section 1](#), for much of the paper we will work in the extra generality of “twisted group actions”. Specifically, let \mathbf{W} be a complete and cocomplete closed symmetric monoidal category, and let G be a group that acts on \mathbf{W} by strong symmetric monoidal functors. We write the action of $g \in G$ on $W \in \mathbf{W}$ as W^g . For simplicity, we suppose that the action is strict, i.e. $(W^g)^h = W^{gh}$ and $W^1 = W$ strictly (and symmetric-monoidal-functorially).

Example 4.1. The case we are most interested in, which will yield our theorems about duality involutions on 2-categories, is when $\mathbf{W} = \mathbf{Cat}$ with G the 2-element group $\{+, -\}$ with $+$ the identity element (a copy of $\mathbb{Z}/2\mathbb{Z}$), and $A^- = A^{\text{op}}$.

However, there are other examples as well. Here are a few, also mentioned in [section 1](#), that yield “duality involutions” with a similar flavor.

Example 4.2. Let $\mathbf{W} = \mathbf{2-Cat}$, with G as the 4-element group $\{++, --, +-, -+\}$ (a copy of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$), and $A^{-+} = A^{\text{op}}$, $A^{+-} = A^{\text{co}}$, and hence $A^{--} = A^{\text{coop}}$. If we give \mathbf{W} the Gray monoidal structure as in [\[GPS95\]](#), this example leads to a theory of duality involutions on Gray-categories.

Example 4.3. Let \mathbf{W} be the category of strict n -categories, with $G = (\mathbb{Z}/2\mathbb{Z})^n$ acting by reversal of k -morphisms at all levels. Since a category enriched in strict n -categories is exactly a strict $(n+1)$ -category, we obtain a theory of duality involutions on strict $(n+1)$ -categories.

Example 4.4. Let $\mathbf{W} = \mathbf{sSet}$, the category of simplicial sets, with $G = \{+, -\}$, and A^- obtained by reversing the directions of all simplices in A . This leads to a theory of duality involutions on simplicially enriched categories that is appropriate when the simplicial sets are regarded as modeling $(\infty, 1)$ -categories as quasi-categories [\[Joy02, Joy, Lur09\]](#), so that simplicially enriched categories are a model for $(\infty, 2)$ -categories. For example, such simplicially enriched categories are used in [\[RV15\]](#) to define a notion of “ ∞ -cosmos” analogous to the “fibrational cosmoi” of [\[Str74\]](#), so such duality involutions could be a first step towards an ∞ -version of [\[Web07\]](#).

Example 4.5. Combining the ideas of the last two examples, if \mathbf{W} is the category of Θ_n -spaces as in [\[Rez10\]](#), then $(\mathbb{Z}/2\mathbb{Z})^n$ acts on it by reversing directions at all dimensions. Thus, we obtain a theory of duality involutions on categories enriched in Θ_n -spaces, which in [\[BR13\]](#) were shown to be a model of $(\infty, n+1)$ -categories.

Note that we do *not* assume the action of G on \mathbf{W} is by \mathbf{W} -enriched functors, since in most of the above examples this is not the case. In particular, $(-)^{\text{op}}$ is not a 2-functor. We also note that most or all of the theory would probably be the same if G were a 2-group rather than just a group, but we do not need this extra generality.

Since the action of G on \mathbf{W} is symmetric monoidal, it extends to an action on $\mathbf{W-Cat}$ applied homwise, which we also write \mathcal{A}^g , i.e. $\mathcal{A}^g(x, y) = (\mathcal{A}(x, y))^g$. In our

motivating example 4.1 we have $\mathcal{A}^- = \mathcal{A}^{\text{co}}$ for a 2-category \mathcal{A} . We now define a 2-monad T on $\mathbf{W}\text{-Cat}$ by

$$T\mathcal{A} = \sum_{g \in G} \mathcal{A}^g.$$

The unit $\mathcal{A} \rightarrow T\mathcal{A}$ includes the summand indexed by $1 \in G$, and the multiplication uses the fact that each action, being an equivalence of categories (indeed, an isomorphism of categories), is cocontinuous:

$$TT\mathcal{A} = \sum_{g \in G} (T\mathcal{A})^g = \sum_{g \in G} \left(\sum_{h \in G} \mathcal{A}^h \right)^g \cong \sum_{g \in G} \sum_{h \in G} (\mathcal{A}^h)^g \cong \sum_{g \in G} \sum_{h \in G} \mathcal{A}^{hg}$$

which we can map into $T\mathcal{A}$ by sending the (g, h) summand to the hg -summand.

We will refer to a normal pseudo T -algebra structure as a **twisted G -action**; it equips a \mathbf{W} -category \mathcal{A} with actions $(-)^g : \mathcal{A}^g \rightarrow \mathcal{A}$ that are suitably associative up to coherent isomorphism (with $(x)^1 = x$ strictly). In our motivating example of $\mathbf{W} = \mathbf{Cat}$ and $G = \{+, -\}$, the monad T agrees with the one we constructed in section 3; thus twisted G -actions are strong duality involutions (and likewise for their morphisms and 2-cells).

Example 4.6. If we write $[x, y]$ for the internal-hom of \mathbf{W} , then we have maps $[x, y]^g \rightarrow [x^g, y^g]$ obtained by adjunction from the composite

$$[x, y]^g \otimes x^g \xrightarrow{\sim} ([x, y] \otimes x)^g \rightarrow y^g$$

Since the $[x, y]$ are the hom-objects of the \mathbf{W} -category \mathbf{W} , these actions assemble into a \mathbf{W} -functor $(-)^g : \mathbf{W}^g \rightarrow \mathbf{W}$, and as g varies they give \mathbf{W} itself a twisted G -action. (Thus, among the three different actions we are denoting by $(-)^g$ — the given one on \mathbf{W} , the induced one on $\mathbf{W}\text{-Cat}$, and an arbitrary twisted G -action — the first is a special case of the third.) In particular, we obtain in this way the canonical strong (in fact, strict) duality involution on \mathbf{Cat} .

Now we note that T extends to a normal monad in the sense of [CS10] on the proarrow equipment $\mathbf{W}\text{-Prof}$, as follows. As in [Shu08, CS10], we view equipments as pseudo double categories satisfying with a “fibrancy” condition saying that horizontal arrows (the “proarrow” direction, for us) can be pulled back universally along vertical ones (the “functor” direction). In $\mathbf{W}\text{-Prof}$ the objects are \mathbf{W} -categories, a horizontal arrow $\mathcal{A} \rightrightarrows \mathcal{B}$ is a profunctor (i.e. a \mathbf{W} -functor $\mathcal{B}^{\text{op}} \otimes \mathcal{A} \rightarrow \mathbf{W}$), a vertical arrow $\mathcal{A} \rightarrow \mathcal{B}$ is a \mathbf{W} -functor, and a square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{M} & \mathcal{B} \\ F \downarrow & \Downarrow & \downarrow G \\ \mathcal{C} & \xrightarrow{N} & \mathcal{D} \end{array}$$

is a \mathbf{W} -natural transformation $M(b, a) \rightarrow N(G(b), F(a))$. A monad on an equipment is strictly functorial in the vertical direction, laxly functorial in the horizontal direction, and its multiplication and unit transformations consist of vertical arrows and squares.

In our case, we already have the action of T on \mathbf{W} -categories and \mathbf{W} -functors. A \mathbf{W} -profunctor $M : \mathcal{A} \rightrightarrows \mathcal{B}$ induces another one $M^g : \mathcal{A}^g \rightrightarrows \mathcal{B}^g$ by applying the G -action objectwise, and by summing up over g we have $TM : T\mathcal{A} \rightrightarrows T\mathcal{B}$. This

is in fact pseudofunctorial on profunctors. Finally, the unit and multiplication are already defined as vertical arrows, and extend to squares in an evident way:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{M} & \mathcal{B} \\ \eta \downarrow & \Downarrow & \downarrow \eta \\ T\mathcal{A} & \xrightarrow{TM} & T\mathcal{B} \end{array} \quad \begin{array}{ccc} TTA & \xrightarrow{TTM} & TT\mathcal{B} \\ \mu \downarrow & \Downarrow & \downarrow \mu \\ T\mathcal{A} & \xrightarrow{TM} & T\mathcal{B} \end{array}$$

Since we have a monad on an equipment, we can define “ T -multicategories” in $\mathbf{W}\text{-Prof}$, which following [CS10] we call *virtual T -algebras*. For our specific monad T , we will refer to virtual T -algebras as **G -variant \mathbf{W} -categories**. Such a gadget is a \mathbf{W} -category \mathcal{A} together with a profunctor $\underline{A} : \mathcal{A} \nrightarrow T\mathcal{A}$, a unit isomorphism $\mathcal{A}(x, y) \xrightarrow{\sim} \underline{A}(\eta(x), y)$, and a composition

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\underline{A}} & T\mathcal{A} & \xrightarrow{T\underline{A}} & TT\mathcal{A} \\ \parallel & & \Downarrow & & \downarrow \mu \\ \mathcal{A} & \xrightarrow{\underline{A}} & T\mathcal{A} & & \end{array}$$

satisfying associativity and unit axioms. If we unravel this explicitly, we see that a G -variant \mathbf{W} -category has a set of objects along with, for each pair of objects x, y and each $g \in G$, a hom-object $\underline{A}^g(x, y) \in \mathbf{W}$, plus units $\mathbb{1} \rightarrow \underline{A}^1(x, x)$ and compositions

$$\underline{A}^g(y, z) \otimes \underline{A}^h(x, y) \rightarrow \underline{A}^{hg}(x, z)$$

satisfying the expected axioms. (Technically, in addition to the hom-objects $\underline{A}^1(x, y)$ it has the hom-objects $\mathcal{A}(x, y)$ that are isomorphic to them, but we may ignore this duplication of data.) We may refer to the elements of $\underline{A}^g(x, y)$ as **g -variant morphisms**. The rule for the variance of composites is easier to remember when written in diagrammatic order: if we denote $\alpha \in \underline{A}^g(x, y)$ by $\alpha : x \xrightarrow{g} y$, then the composite of $x \xrightarrow{g} y \xrightarrow{h} z$ is $x \xrightarrow{gh} z$. (Of course, in our motivating example G is commutative, so the order makes no difference.)

In the specific case of $G = \{+, -\}$ acting on \mathbf{Cat} , we can unravel the definition more explicitly as follows.

Definition 4.7. A **2-category with contravariance** is a G -variant \mathbf{W} -category for $\mathbf{W} = \mathbf{Cat}$ and $G = \{+, -\}$. Thus it consists of

- A collection $\text{ob } \underline{A}$ of objects;
- For each $x, y \in \text{ob } \underline{A}$, a pair of categories $\underline{A}^+(x, y)$ and $\underline{A}^-(x, y)$;
- For each $x \in \text{ob } \underline{A}$, an object $1_x \in \underline{A}^+(x, x)$;
- For each $x, y, z \in \text{ob } \underline{A}$, composition functors

$$\begin{aligned} \underline{A}^+(y, z) \times \underline{A}^+(x, y) &\longrightarrow \underline{A}^+(x, z) \\ \underline{A}^-(y, z) \times \underline{A}^-(x, y)^{\text{op}} &\longrightarrow \underline{A}^+(x, z) \\ \underline{A}^+(y, z) \times \underline{A}^-(x, y) &\longrightarrow \underline{A}^-(x, z) \\ \underline{A}^-(y, z) \times \underline{A}^+(x, y)^{\text{op}} &\longrightarrow \underline{A}^-(x, z); \end{aligned}$$

such that

- Four $(2 \cdot 2^1)$ unitality diagrams commute; and

- Eight (2^3) associativity diagrams commute.

Like any kind of generalized multicategory, G -variant \mathbf{W} -categories form a 2-category. We leave it to the reader to write out explicitly what the morphisms and 2-cells in this 2-category look like; in our example of interest we will call them **2-functors preserving contravariance** and **2-natural transformations respecting contravariance**.

Now, according to [CS10, Theorem 9.2], any twisted G -action $a : T\mathcal{A} \rightarrow \mathcal{A}$ gives rise to a G -variant \mathbf{W} -category with $\underline{\mathbf{A}} = \mathcal{A}(a, 1)$, which in our situation means $\underline{\mathbf{A}}^g(x, y) = \mathcal{A}(x^g, y)$ (where x^g refers, as before, to the g -component of the action $a : T\mathcal{A} \rightarrow \mathcal{A}$). In particular, any 2-category with a strong duality involution can be regarded as a 2-category with contravariance, where $\underline{\mathbf{A}}^+(x, y) = \mathcal{A}(x, y)$ and $\underline{\mathbf{A}}^-(x, y) = \mathcal{A}(x^\circ, y)$.

Moreover, by [CS10, Corollary 9.4], a G -variant \mathbf{W} -category $\underline{\mathbf{A}}$ arises from a twisted G -action exactly when

- (i) The profunctor $\underline{\mathbf{A}} : \mathcal{A} \nrightarrow T\mathcal{A}$ is representable by some $a : T\mathcal{A} \rightarrow \mathcal{A}$, and
- (ii) The induced 2-cell $\bar{a} : a \circ \mu \rightarrow a \circ Ta$ is an isomorphism.

Condition (i) means that for every $x \in \underline{\mathbf{A}}$ and every $g \in G$, there is an object “ x^g ” and an isomorphism $\underline{\mathbf{A}}^g(x, y) \cong \underline{\mathbf{A}}^1(x^g, y)$, natural in y . The Yoneda lemma implies this isomorphism is mediated by a “universal g -variant morphism” $\chi_{g,x} \in \underline{\mathbf{A}}^g(x, x^g)$.

Condition (ii) then means that for any $x \in \underline{\mathbf{A}}$ and $g, h \in G$, the induced map $\psi_{h,g,x} : x^{gh} \rightarrow (x^g)^h$ is an isomorphism. (This map arises by composing $\chi_{g,x} \in \underline{\mathbf{A}}^g(x, x^g)$ with $\chi_{h,x^g} \in \underline{\mathbf{A}}^h(x^g, (x^g)^h)$ to obtain a map in $\underline{\mathbf{A}}^{gh}(x, (x^g)^h)$, then applying the defining isomorphism of x^{gh} .) As usual for generalized multicategories, this is equivalent to requiring a stronger universal property of x^g : that precomposing with $\chi_{g,x}$ induces isomorphisms

$$(4.8) \quad \underline{\mathbf{A}}^h(x^g, y) \xrightarrow{\sim} \underline{\mathbf{A}}^{gh}(x, y)$$

for all $h \in G$. (This again is more mnemonic in diagrammatic notation: any arrow $x \xrightarrow{gh} y$ factors uniquely through $\chi_{g,x}$ by a morphism $x^g \xrightarrow{h} y$, i.e. variances on the arrow can be moved into the action on the domain, preserving order.) This is because the following diagram commutes by definition of $\psi_{h,g,x}$, and the vertical maps are isomorphisms by definition of χ :

$$\begin{array}{ccc} \underline{\mathbf{A}}^{gh}(x, y) & \xleftarrow{-\circ \chi_{g,x}} & \underline{\mathbf{A}}^h(x^g, y) \\ \uparrow -\circ \chi_{gh,x} & & \uparrow -\circ \chi_{h,x^g} \\ \underline{\mathbf{A}}^1(x^{gh}, y) & \xleftarrow{-\circ \psi_{h,g,x}} & \underline{\mathbf{A}}^1((x^g)^h, y) \end{array}$$

If x^g is an object equipped with a morphism $\chi_{g,x} \in \underline{\mathbf{A}}^g(x, x^g)$ satisfying this stronger universal property (4.8), we will call it a **g -variator** of x . In our motivating example $\mathbf{W} = \mathbf{Cat}$ with $g = -$, we call a $-$ -variator an **opposite**. Explicitly, this means the following.

Definition 4.9. In a 2-category with contravariance $\underline{\mathbf{A}}$, a (strict) **opposite** of an object x is an object x° equipped with a contravariant morphism $\chi_x \in \underline{\mathbf{A}}^-(x, x^\circ)$

such that precomposing with χ_x induces isomorphisms of hom-categories for all y :

$$\begin{aligned}\underline{\mathbf{A}}^+(x^\circ, y) &\simeq \underline{\mathbf{A}}^-(x, y) \\ \underline{\mathbf{A}}^-(x^\circ, y) &\simeq \underline{\mathbf{A}}^+(x, y).\end{aligned}$$

In fact, g -variators can also be characterized more explicitly. The second universal property of $\chi_{g,x} \in \underline{\mathbf{A}}^g(x, x^g)$ means in particular that the identity $1_x \in \underline{\mathbf{A}}^1(x, x)$ can be written as $\xi_{g,x} \circ \chi_{g,x}$ for a unique $\xi_{g,x} \in \underline{\mathbf{A}}^{g^{-1}}(x^g, x)$. (This is the first place where we have used the fact that G is a group rather than just a monoid.) Moreover, since

$$(\chi_{g,x} \circ \xi_{g,x}) \circ \chi_{g,x} = \chi_{g,x} \circ (\xi_{g,x} \circ \chi_{g,x}) = \chi_{g,x}$$

it follows by the first universal property of $\chi_{g,x}$ that $\chi_{g,x} \circ \xi_{g,x} = 1_{x^g}$ as well. Thus, $\chi_{g,x}$ and $\xi_{g,x}$ form a “ g -variant isomorphism” between x and x^g .

On the other hand, it is easy to check that any such g -variant isomorphism between x and an object y makes y into a g -variator of x . Thus, we have:

Proposition 4.10. *Any g -variant \mathbf{W} -functor $F : \underline{\mathbf{A}} \rightarrow \underline{\mathbf{B}}$ preserves g -variators. In particular, any 2-functor preserving contravariance also preserves opposites.*

Proof. It obviously preserves “ g -variant isomorphisms”. \square

Thus we have:

Theorem 4.11. *The 2-category of 2-categories with strong duality involutions, duality 2-functors, and duality 2-natural transformations is 2-equivalent to the 2-category of 2-categories with contravariance in which every object has a strict opposite, 2-functors preserving contravariance, and 2-natural transformations respecting contravariance.*

Proof. By [CS10, Theorem 9.13] and the remarks preceding Definition 4.9, the latter 2-category is equivalent to the 2-category of pseudo T -algebras, lax T -morphisms, and T -2-cells. However, Proposition 4.10 implies that in fact every lax T -morphism is a pseudo T -morphism. Finally, every pseudo T -algebra is isomorphic to a normal pseudo one obtained by re-choosing $(-)^1$ to be the identity (which it is assumed to be isomorphic to). \square

5. CONTRAVARIANCE THROUGH ENRICHMENT

We continue with our setup from section 4, with a complete and cocomplete closed monoidal category \mathbf{W} and a group G acting on \mathbf{W} . We start by noticing that the monad T on $\mathbf{W}\text{-Prof}$ constructed in section 4 can actually be obtained in a standard way from a simpler monad.

Recall that there is another equipment $\mathbf{W}\text{-Mat}$ whose objects are sets, whose vertical arrows are functions, and whose horizontal arrows $X \rightarrowtail Y$ are “ \mathbf{W} -valued matrices”, which are just functions $Y \times X \rightarrow \mathbf{W}$; we call them matrices because we compose them by “matrix multiplication”. The equipment $\mathbf{W}\text{-Prof}$ is obtained from $\mathbf{W}\text{-Mat}$ by applying a functor \mathbf{Mod} that constructs monoids (monads) and modules in the horizontal directions. We now observe that our monad T , like many monads on equipments of profunctors, is also in the image of \mathbf{Mod} .

Let S be the following monad on $\mathbf{W}\text{-Mat}$. On objects and vertical arrows, it acts by $S(X) = X \times G$. On a \mathbf{W} -matrix $M : Y \times X \rightarrow \mathbf{W}$ it acts by

$$SM((y, h), (x, g)) = \begin{cases} (M(y, x))^g & g = h \\ \emptyset & g \neq h \end{cases}$$

We may write this schematically using a Kronecker delta as

$$SM((y, h), (x, g)) = \delta_{g,h} \cdot (M(y, x))^g.$$

On a composite of matrices $X \xrightarrow{M} Y \xrightarrow{N} Z$ we have

$$\begin{aligned} (SM \odot SN)((z, k), (x, g)) &= \sum_{(y, h)} (\delta_{g,h} \cdot M(y, x)^g) \otimes (\delta_{h,k} \cdot N(z, y)^h) \\ &\cong \delta_{k,g} \sum_y M(y, x)^g \otimes N(z, y)^g \\ &\cong \delta_{k,g} \left(\sum_y (M(y, x) \otimes N(z, y)) \right)^g \\ &= \delta_{k,g} \cdot (M \odot N)(z, x)^g \\ &= S(M \odot N)((z, k), (x, g)) \end{aligned}$$

making S a pseudofunctor. The monad multiplication and unit are induced from the multiplication and unit of G ; the squares

$$\begin{array}{ccc} X & \xrightarrow{M} & Y \\ \eta \downarrow & \Downarrow & \downarrow \eta \\ SX & \xrightarrow{SM} & SY \end{array} \quad \begin{array}{ccc} SSX & \xrightarrow{SSM} & SSY \\ \mu \downarrow & \Downarrow & \downarrow \mu \\ SX & \xrightarrow{SM} & SY \end{array}$$

map the components $M(y, x)$ and $(M(y, x)^g)^h$ isomorphically to $M(y, x)^1$ and $M(y, x)^{gh}$ respectively.

Now, recalling that $T\mathcal{A} = \sum_{g \in G} \mathcal{A}^g$, we see that $\text{ob}(T\mathcal{A}) = \text{ob}(\mathcal{A}) \times G$ and $T\mathcal{A}((y, h), (x, g)) = \delta_{h,g} \cdot (\mathcal{A}(y, x))^g$, and so in fact $T \cong \text{Mod}(S)$. Thus, by [CS10, Theorem 8.7], virtual T -algebras can be identified with “ S -monoids”; these are defined like virtual S -algebras, with sets and matrices of course replacing categories and profunctors, and omitting the requirement that the unit be an isomorphism. Thus, an S -monoid consists of a set X of objects, a function $\underline{A} : S(X) \times X = X \times G \times X \rightarrow \mathbf{W}$, unit maps $1_x : I \rightarrow \underline{A}^1(x, x)$, and composition maps that turn out to look like $\underline{A}^h(y, z) \otimes \underline{A}^g(x, y) \rightarrow \underline{A}^{gh}(x, z)$. Note that this is exactly what we obtain from a virtual T -algebra by omitting the redundant data of the hom-objects $\mathcal{A}(x, y)$ and their isomorphisms to $\underline{A}^1(x, y)$; this is essentially the content of [CS10, Theorem 8.7] in our case.

In [CS10], the construction of S -monoids is factored into two: first we build a new equipment $\mathbb{H}\text{-Kl}(\mathbf{W}\text{-Mat}, S)$ whose objects and vertical arrows are the same as $\mathbf{W}\text{-Mat}$ but whose horizontal arrows $X \rightarrowtail Y$ are the horizontal arrows $X \rightarrowtail SY$ in $\mathbf{W}\text{-Mat}$, and then we consider horizontal monoids in $\mathbb{H}\text{-Kl}(\mathbf{W}\text{-Mat}, S)$. In fact, $\mathbb{H}\text{-Kl}(\mathbf{W}\text{-Mat}, S)$ is in general only a “virtual equipment” (i.e. we cannot compose its horizontal arrows, though we can “map out of composites” like in a multicategory), but in our case it is an ordinary equipment because S is “horizontally strong” [CS10,

Theorem A.8]. This means that S is a strong functor (which we have already observed) and that the induced maps of matrices

$$\begin{aligned} (\eta, 1)_! M &\rightarrow (1, \eta)^* SM \\ (\mu, 1)_! SSM &\rightarrow (1, \mu)^* \odot SSM \end{aligned}$$

are isomorphisms, where f^* and $f_!$ denote the pullback and its left adjoint push-forward of matrices along functions. Indeed, we have

$$\begin{aligned} (\eta, 1)_! M((y, h), x) &= \delta_{h,1} \cdot M(y, x) && \text{while} \\ (1, \eta)^* SM((y, h), x) &= SM((y, h), (x, 1)) \\ &= \delta_{h,1} \cdot (M(y, x))^1 \\ &= \delta_{h,1} \cdot M(y, x) \end{aligned}$$

and likewise

$$\begin{aligned} (\mu, 1)_! SSM((y, h), ((x, g_1), g_2)) &= \sum_{h_2 h_1 = h} SSM(((y, h_1), h_2), ((x, g_1), g_2)) \\ &= \sum_{h_1 h_2 = h} \delta_{h_2, g_2} \cdot \left(\delta_{h_1, g_1} \cdot M(y, x)^{g_1} \right)^{g_2} \\ &= \sum_{h_1 h_2 = h} \delta_{h_2, g_2} \delta_{h_1, g_1} \cdot M(y, x)^{g_1 g_2} \\ &= \delta_{h, g_1 g_2} \cdot M(y, x)^{g_1 g_2} \end{aligned}$$

while

$$\begin{aligned} (1, \mu)^* \odot SSM((y, h), ((x, g_1), g_2)) &= SSM((y, h), (x, g_1 g_2)) \\ &= \delta_{h, g_1 g_2} \cdot M(y, x)^{g_1 g_2}. \end{aligned}$$

Inspecting the definition of composition in [CS10, Appendix A], we see that the composite of $M : X \rightarrow SY$ and $N : Y \rightarrow SZ$ is

$$(M \odot_S N)((z, h), x) = \sum_y \sum_{g_1 g_2 = h} M((y, g_1), x) \odot N((z, g_2), y)^{g_1}$$

Note that what comes after the \sum_y depends only on $M((y, -), x)$ and $N((z, -), y)$, which are objects of \mathbf{W}^G . Thus, if we write $\int_G \mathbf{W}$ for the category \mathbf{W}^G with the following monoidal structure:

$$(M \otimes N)(h) = \sum_{g_1 g_2 = h} M(g_1) \odot N(g_2)^{g_1}$$

then we have $\mathbb{H}\text{-Kl}(\mathbf{W}\text{-Mat}, S) \cong (\int_G \mathbf{W})\text{-Mat}$. It follows that S -monoids (that is, G -variant \mathbf{W} -categories) can equivalently be regarded as ordinary monoids in the equipment $(\int_G \mathbf{W})\text{-Mat}$. But since monoids in an equipment of matrices are simply enriched categories, we can identify G -variant \mathbf{W} -categories with $\int_G \mathbf{W}$ -enriched categories.

Note that this monoidal structure on $\int_G \mathbf{W}$ is *not* symmetric. It is a version of Day convolution [Day72] that is “twisted” by the action of G on \mathbf{W} (see [Lor16] for further discussion). Like an ordinary Day convolution monoidal structure, it is also closed on both sides (as long as \mathbf{W} is); that is, we have left and right hom-functors \int and \backslash with natural isomorphisms

$$(5.1) \quad (\int_G \mathbf{W})(A \otimes B, C) \cong (\int_G \mathbf{W})(A, B \backslash C) \cong (\int_G \mathbf{W})(B, C \int A).$$

Inspecting the definition of the tensor product in $\int_G \mathbf{W}$, it suffices to define

$$(B \mathbin{\mathbb{N}} C)(g) := \prod_h (B(h)^g \mathbin{\mathbb{N}} C(gh))$$

$$(C \mathbin{\mathbb{L}} A)(g) := \prod_h (C(hg)^{h^{-1}} \mathbin{\mathbb{L}} A(h)^{h^{-1}})$$

(This is another place where we use the fact that G is a group rather than a monoid.) As usual, it follows that $\int_G \mathbf{W}$ can be regarded as a $\int_G \mathbf{W}$ -category (that is, as a g -variant \mathbf{W} -category), with hom-objects $\int_G \mathbf{W}(A, B) := (A \mathbin{\mathbb{N}} B)$. (This fact is often described only for closed *symmetric* monoidal categories, but it works just as well for closed non-symmetric ones, as long as we use the *right* hom.)

Bringing things back down to each a bit, in our specific case with $\mathbf{W} = \mathbf{Cat}$ and $G = \{+, -\}$, let us write $\mathbf{V} = \int_{\{+, -\}} \mathbf{Cat}$. The underlying category of \mathbf{V} is just $\mathbf{Cat} \times \mathbf{Cat}$, but we denote its objects as $A = (A^+, A^-)$, with A^+ the *covariant part* and A^- the *contravariant part*. The monoidal structure on \mathbf{V} is the following nonstandard one:

$$(A \otimes B)^+ := (A^+ \times B^+) \amalg (A^- \times (B^-)^{\text{op}})$$

$$(A \otimes B)^- := (A^+ \times B^-) \amalg (A^- \times (B^+)^{\text{op}})$$

The unit object is

$$\mathbb{1} := (1, 0)$$

where 1 denotes the terminal category and 0 the initial (empty) one. The conclusion of our equipment-theoretic digression above is then the following:

Theorem 5.2. *The 2-category of 2-categories with contravariance, 2-functors preserving contravariance, and 2-natural transformations respecting contravariance is 2-equivalent to the 2-category of \mathbf{V} -enriched categories.* \square

This theorem is easy to prove explicitly as well, of course. A \mathbf{V} -category has, for each pair of objects x, y , a pair of hom-categories $(\underline{\mathbf{A}}^+(x, y), \underline{\mathbf{A}}^-(x, y))$, together with composition functors that end up looking just like those in [Definition 4.7](#), and so on. But I hope that the digression makes this theorem seem less accidental; it also makes it clear how to generalize it to other examples.

The underlying ordinary category $\underline{\mathbf{A}}_o$ of a 2-category $\underline{\mathbf{A}}$ with contravariance, in the usual sense of enriched category theory, consists of its objects and its covariant 1-morphisms (the objects of the categories $\underline{\mathbf{A}}^+(x, y)$). It also has an underlying ordinary 2-category, induced by the lax monoidal forgetful functor $(-)^+ : \mathbf{V} \rightarrow \mathbf{Cat}$, whose hom-categories are the categories $\underline{\mathbf{A}}^+(x, y)$; we denote this 2-category by $\underline{\mathbf{A}}^+$. Of course, there is no 2-category to denote by “ $\underline{\mathbf{A}}^-$ ”, but we could say for instance that $\underline{\mathbf{A}}^-$ is a profunctor from $\underline{\mathbf{A}}^+$ to itself.

6. OPPOSITES THROUGH ENRICHMENT

For most of this section, we let $(\mathbf{V}, \otimes, \mathbb{1})$ be an arbitrary biclosed monoidal category, not assumed symmetric. We are, of course, thinking of our \mathbf{V} from the last section, or more generally $\int_G \mathbf{W}$.

Suppose $\underline{\mathbf{A}}$ is a \mathbf{V} -category, that $x \in \text{ob } \underline{\mathbf{A}}$, and $\omega \in \text{ob } \mathbf{V}$. A **copower** (or **tensor**) of x by ω is an object $\omega \odot x$ of $\underline{\mathbf{A}}$ together with isomorphisms in \mathbf{V} :

$$(6.1) \quad \underline{\mathbf{A}}(\omega \odot x, y) \cong \omega \mathbin{\mathbb{N}} \underline{\mathbf{A}}(x, y)$$

for all $y \in \text{ob } \underline{\mathbf{A}}$, which are \mathbf{V} -natural in the sense that for any $y, z \in \text{ob } \underline{\mathbf{A}}$, the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathbf{A}}(y, z) \otimes \underline{\mathbf{A}}(\omega \odot x, y) & \xrightarrow{\cong} & \underline{\mathbf{A}}(y, z) \otimes (\omega \mathbin{\mathbb{N}} \underline{\mathbf{A}}(x, y)) \longrightarrow \omega \mathbin{\mathbb{N}} (\underline{\mathbf{A}}(y, z) \otimes \underline{\mathbf{A}}(x, y)) \\ \downarrow & & \downarrow \\ \underline{\mathbf{A}}(\omega \odot x, z) & \xrightarrow{\cong} & \omega \mathbin{\mathbb{N}} \underline{\mathbf{A}}(x, z) \end{array}$$

Taking $y = \omega \odot x$ in (6.1), we obtain from $1_{\omega \odot x}$ a canonical map $\omega \rightarrow \underline{\mathbf{A}}(x, \omega \odot x)$, which by the Yoneda lemma determines (6.1) uniquely. Of course, this is just the usual definition of copowers in enriched categories, specialized to enrichment over \mathbf{V} . We have spelled it out explicitly to emphasize that it makes perfect sense even though \mathbf{V} is not symmetric, as long as we choose the correct hom $\mathbin{\mathbb{N}}$ and not $\mathbin{\mathbb{P}}$.

Note that if $\underline{\mathbf{A}} = \underline{\mathbf{V}}$ (the category \mathbf{V} regarded as a \mathbf{V} -category), then the tensor product $\omega \otimes x$ is a copower $\omega \odot x$. Moreover, for general $\underline{\mathbf{A}}$, if $\omega, \varpi \in \mathbf{V}$ and the copowers $\omega \odot x$ and $\varpi \odot (\omega \odot x)$ exist, we have

$$\begin{aligned} \underline{\mathbf{A}}(\varpi \odot (\omega \odot x), y) &\cong \varpi \mathbin{\mathbb{N}} \underline{\mathbf{A}}(\omega \odot x, y) \\ &\cong \varpi \mathbin{\mathbb{N}} (\omega \mathbin{\mathbb{N}} \underline{\mathbf{A}}(x, y)) \\ &\cong (\varpi \otimes \omega) \mathbin{\mathbb{N}} \underline{\mathbf{A}}(x, y) \end{aligned}$$

so that $\varpi \odot (\omega \odot x)$ is a copower $(\varpi \otimes \omega) \odot x$. In particular, these observations mandate writing the copower as $\omega \odot x$ rather than $x \odot \omega$.

Frequently one defines a *power* in a \mathbf{V} -category $\underline{\mathbf{A}}$ to be a copower in $\underline{\mathbf{A}}^{\text{op}}$, but since our \mathbf{V} is not symmetric, \mathbf{V} -categories do not have opposites. Thus, we must define directly a **power** of x by ω to be an object $x \oslash \omega \in \text{ob } \underline{\mathbf{A}}$ together with isomorphisms

$$\underline{\mathbf{A}}(y, x \oslash \omega) \cong \underline{\mathbf{A}}(y, x) \mathbin{\mathbb{P}} \omega$$

for all $y \in \text{ob } \underline{\mathbf{A}}$, which are \mathbf{V} -natural in that the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathbf{A}}(y, x \oslash \omega) \otimes \underline{\mathbf{A}}(z, y) & \xrightarrow{\cong} & (\underline{\mathbf{A}}(y, x) \mathbin{\mathbb{P}} \omega) \otimes \underline{\mathbf{A}}(z, y) \longrightarrow (\underline{\mathbf{A}}(y, x) \otimes \underline{\mathbf{A}}(z, y)) \mathbin{\mathbb{P}} \omega \\ \downarrow & & \downarrow \\ \underline{\mathbf{A}}(z, x \oslash \omega) & \xrightarrow{\cong} & \underline{\mathbf{A}}(z, x) \mathbin{\mathbb{P}} \omega \end{array}$$

Analogous arguments to those for copowers show that when $\underline{\mathbf{A}} = \underline{\mathbf{V}}$, then $x \mathbin{\mathbb{P}} \omega$ is a power $x \oslash \omega$, and that in general we have $(x \oslash \omega) \oslash \varpi \cong x \oslash (\omega \otimes \varpi)$. If both the copower $\omega \odot x$ and the power $x \oslash \omega$ exist, then we have

$$\begin{aligned} \underline{\mathbf{A}}_o(\omega \odot x, y) &\cong \mathbf{V}(1, \underline{\mathbf{A}}(\omega \odot x, y)) \\ &\cong \mathbf{V}(1, \omega \mathbin{\mathbb{N}} \underline{\mathbf{A}}(x, y)) \\ &\cong \mathbf{V}(\omega, \underline{\mathbf{A}}(x, y)) \\ &\cong \mathbf{V}(1, \underline{\mathbf{A}}(x, y) \mathbin{\mathbb{P}} \omega) \\ &\cong \mathbf{V}(1, \underline{\mathbf{A}}(x, y \oslash \omega)) \\ &\cong \underline{\mathbf{A}}_o(x, y \oslash \omega). \end{aligned}$$

so that the endofunctors $(\omega \odot -)$ and $(- \oslash \omega)$ on the underlying 1-category $\underline{\mathbf{A}}_o$ are adjoint. They are *not* adjoint \mathbf{V} -functors, even when $\underline{\mathbf{A}} = \underline{\mathbf{V}}$: in our motivating

example, the isomorphisms (5.1) do not even lift from the 1-category $\underline{\mathbf{V}}_o = \mathbf{V}$ to the 2-category $\underline{\mathbf{V}}^+$.

Now suppose that ω is *right dualizable* in \mathbf{V} , i.e. that we have an object $\omega^* \in \mathbf{V}$ and morphisms $\omega^* \otimes \omega \rightarrow \mathbb{1}$ and $\mathbb{1} \rightarrow \omega \otimes \omega^*$ satisfying the triangle identities. Then $(-\otimes\omega^*)$ is right adjoint to $(-\otimes\omega)$, hence isomorphic to $(\omega \backslash -)$; and dually we have $(\omega \otimes -) \cong (- \int \omega^*)$. Thus, a copower $\omega \odot x$ in a \mathbf{V} -category $\underline{\mathbf{A}}$ is equivalently characterized by an isomorphism

$$(6.2) \quad \underline{\mathbf{A}}(\omega \odot x, -) \cong \underline{\mathbf{A}}(x, -) \otimes \omega^*,$$

while a power $x \oslash \omega^*$ is characterized by an isomorphism

$$(6.3) \quad \underline{\mathbf{A}}(-, x \oslash \omega^*) \cong \omega \otimes \underline{\mathbf{A}}(-, x).$$

However, for fixed x , the right-hand sides of (6.2) and (6.3) are adjoint in the bicategory of \mathbf{V} -modules. Since $\underline{\mathbf{A}}(\omega \odot x, -)$ always has an adjoint $\underline{\mathbf{A}}(-, \omega \odot x)$, and likewise $\underline{\mathbf{A}}(-, x \oslash \omega^*)$ always has an adjoint $\underline{\mathbf{A}}(x \oslash \omega^*, -)$, it follows that giving a copower $\omega \odot x$ is equivalent to giving a power $x \oslash \omega^*$.

Now let us specialize to the case of $\int_G \mathbf{W}$. Then for any $g \in G$, we have a **twisted unit** $\mathbb{1}^g \in \int_G \mathbf{W}$, defined by $\mathbb{1}^g(h) = \delta_{g,h} \cdot \mathbb{1}$. By definition of \backslash and \int , we have

$$\begin{aligned} (\mathbb{1}^h \backslash \underline{\mathbf{A}}(x, y))(g) &\cong \underline{\mathbf{A}}^{gh}(x, y) \quad \text{and} \\ (\underline{\mathbf{A}}(x, y) \int \mathbb{1}^h)(g) &\cong (\underline{\mathbf{A}}^{hg}(x, y))^{h^{-1}}. \end{aligned}$$

Thus, $\mathbb{1}^h \odot x$, if it exists, is characterized by isomorphisms

$$\underline{\mathbf{A}}^g(\mathbb{1}^h \odot x, y) \cong \underline{\mathbf{A}}^{gh}(x, y)$$

that are suitably and jointly natural in y . In other words, a copower $\mathbb{1}^h \odot x$ is precisely an *h-variator* of x as defined in section 4. And in our particular case of $\mathbf{W} = \mathbf{Cat}$, a copower $\mathbb{1}^- \odot x$ is precisely an *opposite* of x as defined in Definition 4.9. Thus we have:

Theorem 6.4. *A 2-category with contravariance has opposites, as in Definition 4.9, exactly if when regarded as a \mathbf{V} -category it has all copowers by $\mathbb{1}^-$. \square*

Note that since $\mathbb{1}^h \otimes \mathbb{1}^{h^{-1}} \cong \mathbb{1}$, in particular $\mathbb{1}^h$ is dualizable. Thus, copowers by $\mathbb{1}^h$ are equivalent to powers by $\mathbb{1}^{h^{-1}}$. In particular, since $- \in \{+, -\}$ is its own inverse, it follows that $\mathbb{1}^-$ is self-dual, and opposites are also characterized by isomorphisms

$$\underline{\mathbf{A}}^+(y, x^\circ) \cong \underline{\mathbf{A}}^-(y, x)^{\text{op}} \quad \text{and} \quad \underline{\mathbf{A}}^-(y, x^\circ) \cong \underline{\mathbf{A}}^+(y, x)^{\text{op}}.$$

This gives another reason why a 2-functor preserving contravariance must preserve opposites: copowers by a dualizable object are absolute colimits [Str83].

7. BICATEGORIES WITH CONTRAVARIANCE

We have now reached the top of the right-hand side of the ladder from section 1. It remains to move across to the other side and head down, starting with a bicategorical version of \mathbf{V} -categories for our $\mathbf{V} = \int_{\{+, -\}} \mathbf{Cat}$.

In fact, it will be convenient to stay in a more general setting. Thus, suppose that our monoidal category \mathbf{W} is actually a 2-category \mathcal{W} , and that our group G acts on it by 2-functors. In this case, the construction of $\int_G \mathbf{W}$ can all be done with 2-categories, obtaining a monoidal 2-category $\int_G \mathcal{W}$ (in the strict sense of

a monoidal **Cat**-enriched category). Since a monoidal 2-category is *a fortiori* a monoidal bicategory, we can consider $\int_G \mathcal{W}$ -enriched bicategories, which we call **g -variant \mathcal{W} -bicategories**.

The most comprehensive extant reference on enriched bicategories seems to be [GS16]. The definition of an enriched bicategory is quite simple: we just write out the definition of bicategory and replace all hom-categories by objects of $\int_G \mathcal{W}$, cartesian products of categories by \otimes , and functors and natural transformations by morphisms and 2-cells in $\int_G \mathcal{W}$. If we write this out explicitly, it consists of the following.

- A collection $\text{ob } \underline{\mathbf{A}}$ of objects;
- For each $x, y \in \text{ob } \underline{\mathbf{A}}$ and $g \in G$, a category $\underline{\mathbf{A}}^g(x, y)$;
- For each $x \in \text{ob } \underline{\mathbf{A}}$, a unit morphism $1_x : \mathbb{1} \rightarrow \underline{\mathbf{A}}^1(x, x)$;
- For each $x, y, z \in \text{ob } \underline{\mathbf{A}}$ and $g, h \in G$, composition morphisms

$$\underline{\mathbf{A}}^h(y, z) \otimes (\underline{\mathbf{A}}^g(x, y))^h \rightarrow \underline{\mathbf{A}}^{gh}(x, z)$$

- For each $x, y \in \text{ob } \underline{\mathbf{A}}$ and $g \in G$, two natural unitality isomorphisms;
- For each $x, y, z, w \in \text{ob } \underline{\mathbf{A}}$ and $g, h, k \in G$, an associativity isomorphism;
- For each $x, y, z \in \text{ob } \underline{\mathbf{A}}$ and $g, h \in G$, a unitality axiom holds; and
- For each $x, y, z, w, u \in \text{ob } \underline{\mathbf{A}}$ and $g, h, k, \ell \in G$, an associativity pentagon holds.

Enriched bicategories, of course, come naturally with a notion of enriched functor. (In fact, as described in [GS16] we have a whole tricategory of enriched bicategories, but we will not need the higher structure.) Explicitly, a $\int_G \mathcal{W}$ -enriched functor $F : \underline{\mathbf{A}} \rightarrow \underline{\mathbf{B}}$ consists of

- A function $F : \text{ob } \underline{\mathbf{A}} \rightarrow \text{ob } \underline{\mathbf{B}}$; and
- For each $x, y \in \text{ob } \underline{\mathbf{A}}$ and $g \in G$, a morphism $F : \underline{\mathbf{A}}^g(x, y) \rightarrow \underline{\mathbf{B}}^g(Fx, Fy)$;
- For each $x \in \text{ob } \underline{\mathbf{A}}$, an isomorphism $F(1_x) \cong 1_{Fx}$;
- For each $x, y, z \in \text{ob } \underline{\mathbf{A}}$ and $g, h \in G$, a natural functoriality isomorphism of the form $(Fg)(Ff) \cong F(gf)$;
- For each $x, y \in \text{ob } \underline{\mathbf{A}}$ and $g \in G$, a unit coherence diagram commutes;
- For each $x, y, z, w \in \text{ob } \underline{\mathbf{A}}$ and $g, h, k \in G$, an associativity coherence diagram commutes.

In the case of interest, we have $\mathbf{W} = \mathbf{Cat}$, which is of course a 2-category. However, as we have remarked, $(-)^{\text{op}}$ is not a 2-functor on \mathcal{Cat} , and so $\{+, -\}$ does not act on \mathcal{Cat} through 2-functors. However, $(-)^{\text{op}}$ is a 2-functor on \mathcal{Cat}_g , the 2-category of categories, functors, and natural *isomorphisms*; so this is what we take as our \mathcal{W} . We denote the resulting monoidal 2-category $\int_G \mathcal{W}$ by \mathcal{V} , and make the obvious definition:

Definition 7.1. A **bicategory with contravariance** is a \mathcal{V} -enriched bicategory, and a **pseudofunctor preserving contravariance** is a \mathcal{V} -enriched functor.

If we write this out explicitly in terms of covariant and contravariant parts, we see that a bicategory with contravariance has four kinds of composition functors, eight kinds of associativity isomorphisms, and sixteen coherence pentagons. Working with an abstract \mathcal{W} and G thus allows us to avoid tedious case-analyses.

We now generalize the enriched notion of g -variator (and hence of “opposite”) from section 6 to the bicategorical case. For any $\omega \in \int_G \mathcal{W}$, any $\int_G \mathcal{W}$ -bicategory $\underline{\mathbf{A}}$, and any $x \in \text{ob } \underline{\mathbf{A}}$, a **copower** of x by ω is an object $\omega \odot x$ together with a map $\omega \rightarrow \underline{\mathbf{A}}(x, \omega \odot x)$ such that for any y the induced map $\underline{\mathbf{A}}(\omega \odot x, y) \rightarrow \omega \searrow \underline{\mathbf{A}}(x, y)$

is an *equivalence* (not necessarily an isomorphism). (This is essentially the special case of [GS16, 10.1] when \mathcal{B} is the unit $\int_G \mathcal{W}$ -bicategory.)

As in section 6, we are mainly interested in the case when ω is one of the twisted units $\mathbb{1}^g$. In this case we again write x^g for $\mathbb{1}^g \odot x$, and the map $\mathbb{1}^g \rightarrow \underline{\mathbb{A}}(x, x^g)$ is just a g -variant morphism $\chi_{g,x} \in \underline{\mathbb{A}}^g(x, x^g)$. Its universal property says that any gh -variant morphism $x \xrightarrow{gh} y$ factors essentially uniquely through $\chi_{g,x}$ via an h -variant morphism $x^g \xrightarrow{h} y$ (and similarly for 2-cells); that is, we have equivalences

$$\underline{\mathbb{A}}^h(x^g, y) \simeq \underline{\mathbb{A}}^{gh}(x, y)$$

As before, by Yoneda arguments this is equivalent to having a g -variant morphism $x \xrightarrow{g} x^g$ and a g^{-1} -variant morphism $x^g \xrightarrow{g^{-1}} x$ whose composites in both directions are isomorphic to identities; that is, a “ g -variant equivalence”. In the specific example of $\mathcal{W} = \mathcal{Cat}_g$ and $G = \{+, -\}$, we of course call x^- a **(weak) opposite** of x , written x° .

Our goal now is to show that any weak duality involution on a bicategory \mathcal{A} gives it the structure of a bicategory with contravariance having weak opposites; but to minimize case analyses, we will work in the generality of \mathcal{W} and G . Thus, we first define a **(weak, strictly unital) twisted G -action** on a \mathcal{W} -category \mathcal{A} to consist of:

- For each $g \in G$, a \mathcal{W} -functor $(-)^g : \mathcal{A}^g \rightarrow \mathcal{A}$. (Note that here \mathcal{A}^g denotes the hom-wise action, $\mathcal{A}^g(x, y) = \mathcal{A}(x, y)^g$.) When $g = 1$ is the unit element of G , we ask that $(-)^1$ be exactly equal to the identity functor.
- For each $g, h \in G$, a \mathcal{W} -pseudonatural adjoint equivalence

$$\begin{array}{ccc} (\mathcal{A})^{gh} & \xrightarrow{(-)^{gh}} & \mathcal{A} \\ & \Downarrow \eta & \nearrow (-)^h \\ ((-)^g)^h & \xrightarrow{\quad} & \mathcal{A}^h \end{array}$$

(Note that $((-)^g)^h$ means the homwise endofunctor $(-)^h$ of \mathcal{W} -bicategories applied to the action functor $(-)^g : \mathcal{A}^g \rightarrow \mathcal{A}$.) When g or h is $1 \in G$, we ask that η be exactly the identity transformation.

- For each $g, h, k \in G$, an invertible \mathcal{W} -modification

$$\begin{array}{ccc} & \mathcal{A}^{hk} & \\ \nearrow & \downarrow & \searrow \\ \mathcal{A}^{ghk} & & \mathcal{A} \\ \searrow & \uparrow & \nearrow \\ & \mathcal{A}^k & \end{array} \xRightarrow{\zeta} \begin{array}{ccc} & \mathcal{A}^{hk} & \\ \nearrow & \eta_{g,hk} & \searrow \\ \mathcal{A}^{ghk} & \xrightarrow{\quad} & \mathcal{A} \\ \searrow & \eta_{gh,k} & \nearrow \\ & \mathcal{A}^k & \end{array}$$

As before, when g, h , or k is $1 \in G$, we ask that ζ be exactly the identity.

- For each $g, h, k, \ell \in G$, a 4-simplex diagram of instances of ζ commutes.

In our motivating example of $\mathcal{W} = \mathcal{Cat}_g$ and $G = \{+, -\}$, the strict identity requirements mean that:

- The only nontrivial action is $(-)^-$, which we write as $(-)^\circ$.

- The only nontrivial η is $\eta_{-, -}$, which has the same type as the η in Definition 2.1.
- The only nontrivial ζ is $\zeta_{-, -, -}$, which has an equivalent type to the ζ in Definition 2.1 (since $-- = +$ is the identity, $\eta_{-, -}$ and $\eta_{-, -}$ are identities, so the type of ζ displayed above has moved one copy of η from the codomain to the domain).
- The only nontrivial axiom likewise reduces to the one given in Definition 2.1.

Thus, this really does generalize our notion of duality involution. Now we will show:

Theorem 7.2. *Let \mathcal{A} be a \mathcal{W} -bicategory with a twisted G -action, and for $x, y \in \mathcal{A}$ and $g \in G$ define $\underline{\mathcal{A}}^g(x, y) = \mathcal{A}(x^g, y)$. Then $\underline{\mathcal{A}}$ is a $\int_G \mathcal{W}$ -bicategory with copowers by all the twisted units.*

Proof. We define the composition morphisms as follows:

$$\begin{aligned}
 \underline{\mathcal{A}}^h(y, z) \otimes (\underline{\mathcal{A}}^g(x, y))^h &= \mathcal{A}(y^h, z) \otimes (\mathcal{A}(x^g, y))^h \\
 &= \mathcal{A}(y^h, z) \otimes \mathcal{A}^h(x^g, y) \\
 &\xrightarrow{(-)^h} \mathcal{A}(y^h, z) \otimes \mathcal{A}((x^g)^h, y^h) \\
 &\xrightarrow{\text{comp}} \mathcal{A}((x^g)^h, z) \\
 &\xrightarrow{- \circ \eta_{g, h}} \mathcal{A}(x^{gh}, z) \\
 &= \underline{\mathcal{A}}^{gh}(x, z)
 \end{aligned}$$

Informally (or, formally, in an appropriate internal linear type theory of \mathcal{W}), we can say that the composite of $\beta \in \underline{\mathcal{A}}^h(y, z)$ and $\alpha \in (\underline{\mathcal{A}}^g(x, y))^h$ is

$$\beta \circ \alpha^h \circ \eta_{g, h}$$

where \circ denotes composition in \mathcal{A} . Expressed in the same way, the associator for $\alpha \in (\underline{\mathcal{A}}^g(x, y))^{hk}$, $\beta \in (\underline{\mathcal{A}}^h(y, z))^k$, and $\gamma \in \underline{\mathcal{A}}^k(z, w)$ is

$$\begin{aligned}
 (\gamma \circ \beta^k \circ \eta_{h, k}) \circ \alpha^{hk} \circ \eta_{g, hk} &\cong \gamma \circ \beta^k \circ (\alpha^h)^k \circ \eta_{h, k} \circ \eta_{g, hk} \\
 &\cong \gamma \circ \beta^k \circ (\alpha^h)^k \circ \eta_{g, h}^k \circ \eta_{gh, k} \\
 &\cong \gamma \circ (\beta \circ \alpha^h \circ \eta_{g, h})^k \circ \eta_{gh, k}
 \end{aligned}$$

using the naturality of η , the modification ζ , and the functoriality of $(-)^k$ (and omitting the associativity isomorphisms of \mathcal{A} , by coherence for bicategories).

For the unit, since $\underline{\mathcal{A}}^1(x, y) = \mathcal{A}(x, y)$, the unit map $\mathbb{1} \rightarrow \underline{\mathcal{A}}^1(x, y)$ is just the unit of \mathcal{A} . One unit isomorphism is just that of \mathcal{A} , while the other is that of \mathcal{A} together with the unit isomorphism of the pseudofunctor $(-)^g$. And the associator appearing in the unit axiom has $g = k = 1$, so all the η 's collapse and it is essentially trivial, and the unit axiom follows immediately from that of \mathcal{A} .

To show that $\underline{\mathcal{A}}$ is a $\int_G \mathcal{W}$ -bicategory, it remains to consider the pentagon axiom. Omitting \circ from now on, the pentagon axiom is an equality of two morphisms

$$\delta \gamma^\ell \eta_{k, \ell} \beta^{k\ell} \eta_{h, k\ell} \alpha^{hk\ell} \eta_{g, hk\ell} \longrightarrow \delta (\gamma (\beta \alpha^h \eta_{g, h})^k \eta_{gh, k})^\ell \eta_{ghk, \ell}$$

By naturality of the functoriality isomorphisms for the actions $(-)^g$, we can certainly push all applications of them to the end where they will be equal; thus it suffices to compare the morphisms

$$\delta \gamma^\ell \eta_{k, \ell} \beta^{k\ell} \eta_{h, k\ell} \alpha^{hk\ell} \eta_{g, hk\ell} \longrightarrow \delta \gamma^\ell (\beta^k)^\ell ((\alpha^h)^k)^\ell (\eta_{g, h}^k)^\ell \eta_{gh, k}^\ell \eta_{ghk, \ell}$$

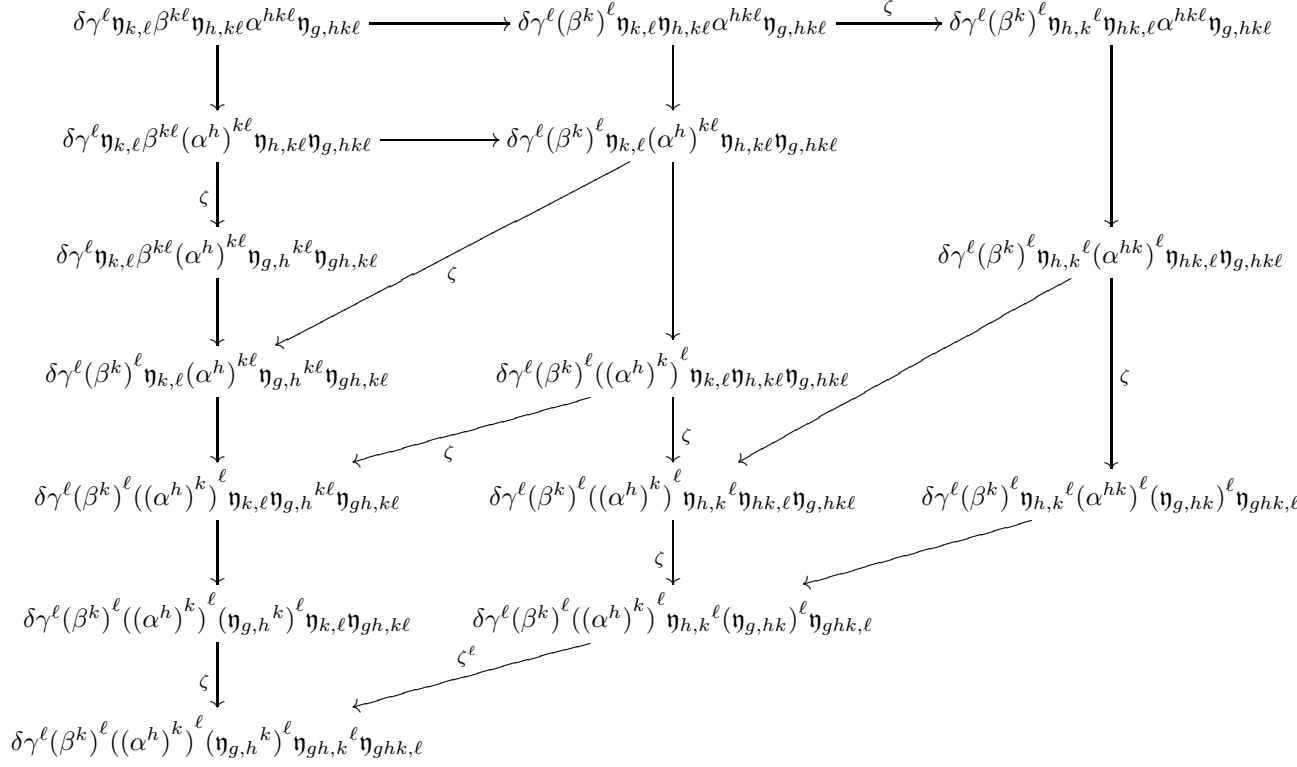


FIGURE 1. The pentagon axiom

This is done in [Figure 1](#), where most of the regions are naturality, except for the one at the bottom left which is the 4-simplex axiom for ζ .

Now we must show that $\underline{\mathbf{A}}$ has copowers by the twisted units; of course we will use x^g as the copower $\mathbb{1}^g \odot x$. Since $\underline{\mathbf{A}}^g(x, x^g) = \mathcal{A}(x^g, x^g)$ by definition, for $\chi_{g,x}$ we can take the identity map of x^g in \mathcal{A} . By definition of composition in $\underline{\mathbf{A}}$, the induced precomposition map

$$\underline{\mathbf{A}}^h(x^g, y) \rightarrow \underline{\mathbf{A}}^{gh}(x, y)$$

is essentially just precomposition with η :

$$\mathcal{A}((x^g)^h, y) \rightarrow \mathcal{A}(x^{gh}, y)$$

and hence is an equivalence. Thus, $\underline{\mathbf{A}}$ has copowers by the twisted units. \square

Inspecting the construction, we also conclude:

Scholium 7.3. *If \mathcal{A} has a twisted G -action in the sense of [section 4](#), regarded as having a weak twisted G -action in the sense defined above with the actions strict functors, η strictly natural, and ζ an identity, then the $\int_G \mathcal{W}$ -bicategory constructed in [Theorem 7.2](#) is actually a strict $\int_G \mathbf{W}$ -category, and this construction agrees with the one in [§4–5](#). In particular, if \mathcal{A} is a 2-category with a strong duality involution,*

and we regard it as a bicategory with a weak duality involution to construct a bicategory with contravariance $\underline{\mathcal{A}}$, the result is the 2-category with contravariance we already obtained from it in [section 4](#). \square

With some more work we could enhance [Theorem 7.2](#) to a whole equivalence of tricategories. However, all we will need for our coherence theorem, in addition to [Theorem 7.2](#) and [Scholium 7.3](#), is to go backwards on biequivalences.

Before stating such a theorem, we have to define what we want to get out of it. Suppose \mathcal{A} and \mathcal{B} are \mathcal{W} -bicategories with twisted G -action; by a **twisted G -functor** $F : \mathcal{A} \rightarrow \mathcal{B}$ we mean a functor of \mathcal{W} -bicategories together with:

- For each $g \in G$, a \mathcal{W} -pseudonatural adjoint equivalence

$$\begin{array}{ccc} \mathcal{A}^g & \xrightarrow{F^g} & \mathcal{B}^g \\ (-)^g \downarrow & \Downarrow_i & \downarrow (-)^g \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

- For each $g, h \in G$, an invertible \mathcal{W} -modification

$$\begin{array}{ccc} & \mathcal{A}^{gh} & \xrightarrow{F^{gh}} \mathcal{B}^{gh} \\ & \swarrow ((-)^g)^h & \Downarrow_i \swarrow ((-)^g)^h \\ \mathcal{A}^h & \xrightarrow{F^h} \mathcal{B}^h & \\ (-)^h \downarrow & \Downarrow_i \downarrow (-)^h & \nearrow \eta_{gh} \nearrow (-)^{gh} \\ \mathcal{A} & \xrightarrow{F} \mathcal{B} & \end{array} \quad \xRightarrow{\theta} \quad \begin{array}{ccc} & \mathcal{A}^{gh} & \xrightarrow{F^{gh}} \mathcal{B}^{gh} \\ & \swarrow ((-)^g)^h & \Downarrow_i \swarrow ((-)^g)^h \\ \mathcal{A}^h & \xrightarrow{F^h} \mathcal{B}^h & \\ (-)^h \downarrow & \Downarrow_i \downarrow (-)^h & \nearrow \eta_{gh} \nearrow (-)^{gh} \\ \mathcal{A} & \xrightarrow{F} \mathcal{B} & \end{array}$$

(As before, $((-)^g)^h$ denotes the functorial action of the homwise endofunctor $(-)^h$ of \mathcal{W} -bicategories on the given action functor $(-)^g : \mathcal{A}^h \rightarrow \mathcal{A}$.) This can be written formally as

$$i_h \circ i_g^h \circ \eta_{g,h}^{\mathcal{B}} \cong \eta_{g,h}^{\mathcal{A}} \circ i_{gh}$$

- For all $g, h, k \in G$, an axiom holds that can be written formally as the commutative diagram shown in [Figure 2](#).

Theorem 7.4. *Suppose \mathcal{A} and \mathcal{B} are \mathcal{W} -bicategories with twisted G -action, with resulting $\int_G \mathcal{W}$ -bicategories $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$. If $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ are biequivalent as $\int_G \mathcal{W}$ -bicategories, then \mathcal{A} and \mathcal{B} are biequivalent by a twisted G -functor.*

In particular, if two bicategories \mathcal{A} and \mathcal{B} with duality involution give rise to biequivalent bicategories-with-contravariance, then \mathcal{A} and \mathcal{B} are biequivalent by a duality pseudofunctor.

Proof. Let $F : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{B}}$ be a $\int_G \mathcal{W}$ -biequivalence. In particular, then, it is a biequivalence on the 1-parts, hence a biequivalence $\mathcal{A} \simeq \mathcal{B}$.

Now by [Theorem 7.2](#), for any $x \in \mathcal{A}$ we have a “ g -variant equivalence” $x \xrightarrow{g} x^g$ with inverse $x^g \xrightarrow{g^{-1}} x$. This structure is preserved by F , so we have a g -variant equivalence between Fx and $F(x^g)$. But we also have a g -variant equivalence

$$\begin{array}{ccccc}
i_k i_h^k (i_g^h)^k \eta_{h,k}^{\mathcal{B}} \eta_{g,hk}^{\mathcal{B}} & \longrightarrow & i_k i_h^k \eta_{h,k}^{\mathcal{B}} i_g^{hk} \eta_{g,hk}^{\mathcal{B}} & \xrightarrow{\theta} & \eta_{h,k}^{\mathcal{A}} i_{hk} i_g^{hk} \eta_{g,hk}^{\mathcal{B}} \\
\downarrow \zeta & & & & \downarrow \theta \\
i_k i_h^k (i_g^h)^k \eta_{g,h}^{\mathcal{B}} \eta_{gh,k}^{\mathcal{B}} & & & & \eta_{h,k}^{\mathcal{A}} \eta_{g,hk}^{\mathcal{A}} i_{ghk} \\
\downarrow & & & & \downarrow \zeta \\
i_k (i_h i_g^h \eta_{g,h}^{\mathcal{B}})^k \eta_{gh,k}^{\mathcal{B}} & & & & (\eta_{g,h}^{\mathcal{A}})^k \eta_{gh,k}^{\mathcal{A}} i_{ghk} \\
\downarrow \theta & & & & \uparrow \theta \\
i_k (\eta_{g,h}^{\mathcal{A}} i_{gh})^k \eta_{gh,k}^{\mathcal{B}} & \longrightarrow & i_k (\eta_{g,h}^{\mathcal{A}})^k (i_{gh})^k \eta_{gh,k}^{\mathcal{B}} & \longrightarrow & (\eta_{g,h}^{\mathcal{A}})^k i_k (i_{gh})^k \eta_{gh,k}^{\mathcal{B}}
\end{array}$$

FIGURE 2. The axiom for θ

between Fx and $(Fx)^g$, and composing them we obtain an ordinary (1-variant) isomorphism $(Fx)^g \cong F(x^g)$. These supply the components of i ; their pseudo-naturality is straightforward.

Now, by construction of the copowers by twisted units, it follows that $\eta_{g,h} : x^{gh} \rightarrow (x^g)^h$ is isomorphic to the composite of the variant equivalences

$$x^{gh} \xrightarrow{(gh)^{-1}} x \xrightarrow{g} x^g \xrightarrow{h} (x^g)^h$$

while ζ is obtained by canceling and uncanceled some of these equivalences. In particular, when η is composed with i , we can simply cancel some inverse variant equivalences to obtain the components of θ . As for Figure 2, its source is

$$\begin{aligned}
(F(x))^{ghk} & \xrightarrow{(ghk)^{-1}} F(x) \xrightarrow{g} (F(x))^g \xrightarrow{hk} ((F(x))^g)^{hk} \\
& \xrightarrow{(hk)^{-1}} (F(x))^g \xrightarrow{h} ((F(x))^g)^h \xrightarrow{k} (((F(x))^g)^h)^k \\
& \xrightarrow{g^{-1}} ((F(x))^h)^k \xrightarrow{g} ((F(x^g))^h)^k \\
& \xrightarrow{h^{-1}} (F(x^g))^k \xrightarrow{h} (F((x^g)^h))^k \xrightarrow{k^{-1}} F((x^g)^h) \xrightarrow{k} F(((x^g)^h)^k)
\end{aligned}$$

while its target is

$$\begin{aligned}
(F(x))^{ghk} & \xrightarrow{(ghk)^{-1}} F(x) \xrightarrow{ghk} F(x^{ghk}) \\
& \xrightarrow{(ghk)^{-1}} F(x) \xrightarrow{gh} F(x^{gh}) \xrightarrow{k} F((x^{gh})^k) \\
& \xrightarrow{(gh)^{-1}} F(x^k) \xrightarrow{g} F((x^g)^k) \xrightarrow{h} F(((x^g)^h)^k)
\end{aligned}$$

Here we have applied functors such as $(-)^k$ to variant morphisms; we can define this by simply “conjugating” with the variant equivalences $x \xrightarrow{k} x^k$. We leave it to the reader to apply naturality and cancel all the redundancy in these composites,

reducing them both to

$$(F(x))^{ghk} \xrightarrow{(ghk)^{-1}} F(x) \xrightarrow{g} F(x^g) \xrightarrow{h} F((x^g)^h) \xrightarrow{k} F(((x^g)^h)^k)$$

so that they are equal. \square

Therefore, to strictify a bicategory with duality involution, it will suffice to strictify its corresponding bicategory with contravariance. This is the task of the next, and final, section.

8. COHERENCE FOR ENRICHED BICATEGORIES

We could continue in the generality of G and \mathcal{W} , but there seems little to be gained by it any more.

Theorem 8.1. *Any bicategory with contravariance is biequivalent to a 2-category with contravariance.*

Proof. Just as there are two ways to prove the coherence theorem for ordinary bicategories, there are two ways to prove this coherence theorem. The first is an algebraic one, involving formally adding strings of composable arrows that hence compose strictly associatively. This can be expressed abstractly using the same general coherence theorem for pseudo-algebras over a 2-monad that we used in [section 3](#). As sketched at the end of [\[Shu12, §4\]](#), this theorem (or a slightly generalization of it) applies as soon as we observe that our 2-category \mathcal{V} is closed monoidal and cocomplete.

The other method is by a Yoneda embedding. To generalize this to the enriched (and non-symmetric) case, first note that for any \mathcal{V} -bicategory $\underline{\mathbf{A}}$, by [\[GS16, 9.3–9.6\]](#) we have a \mathcal{V} -bicategory $\mathcal{M}\underline{\mathbf{A}}$ of *moderate $\underline{\mathbf{A}}$ -modules*, and a Yoneda embedding $\underline{\mathbf{A}} \rightarrow \mathcal{M}\underline{\mathbf{A}}$ that is fully faithful. Thus, $\underline{\mathbf{A}}$ is biequivalent to its image in $\mathcal{M}\underline{\mathbf{A}}$. However, since \mathcal{V} is a strict 2-category that is closed and complete, $\mathcal{M}\underline{\mathbf{A}}$ is actually a strict \mathcal{V} -category, and hence so is any full subcategory of it.

Explicitly, an $\underline{\mathbf{A}}$ -module consists of categories $F^+(x)$ and $F^-(x)$ for each $x \in \underline{\mathbf{A}}$ together with actions

$$\begin{aligned} F^+(y) \times \underline{\mathbf{A}}^+(x, y) &\rightarrow F^+(x) \\ F^+(y) \times \underline{\mathbf{A}}^-(x, y) &\rightarrow F^-(x) \\ F^-(y) \times \underline{\mathbf{A}}^+(x, y)^{\text{op}} &\rightarrow F^-(x) \\ F^-(y) \times \underline{\mathbf{A}}^-(x, y)^{\text{op}} &\rightarrow F^+(x) \end{aligned}$$

and coherent associativity and unitality isomorphisms. A covariant $\underline{\mathbf{A}}$ -module morphism consists of functors $F^+(x) \rightarrow G^+(x)$ and $F^-(x) \rightarrow G^-(x)$ that commute up to coherent natural isomorphism with the actions, while a contravariant one consists similarly of functors $F^+(x)^{\text{op}} \rightarrow G^-(x)$ and $F^-(x)^{\text{op}} \rightarrow G^+(x)$. Since $\mathcal{C}at$ is a strict 2-category, the bicategory-with-contravariance of modules is in fact a strict 2-category with contravariance. The Yoneda embedding, of course, sends each $z \in \underline{\mathbf{A}}$ to the representable Y_z defined by $Y_z^+(x) := \underline{\mathbf{A}}^+(x, z)$ and $Y_z^-(x) := \underline{\mathbf{A}}^-(x, z)$. \square

We have almost completed our trip over the ladder; it remains to make the following observation and then put all the pieces together.

Theorem 8.2. *If $\underline{\mathbf{A}}$ is a 2-category with contravariance that has weak opposites, then it is biequivalent to a 2-category with contravariance having strict opposites.*

Proof. Let $\underline{\mathbf{A}}'$ be the free cocompletion of $\underline{\mathbf{A}}$, as a strict \mathcal{V} -category, under strict opposites (a strict \mathcal{V} -weighted colimit). It is easy to see that this can be done in one step, by considering the collection of all opposites of representables in the presheaf \mathcal{V} -category of $\underline{\mathbf{A}}$. Thus, the embedding $\underline{\mathbf{A}} \rightarrow \underline{\mathbf{A}}'$ is \mathcal{V} -fully-faithful, and every object of $\underline{\mathbf{A}}'$ is the strict opposite of something in the image. However, $\underline{\mathbf{A}}$ has weak opposites, which are preserved by any \mathcal{V} -functor, and any strict opposite is a weak opposite. Thus, every object of $\underline{\mathbf{A}}'$ is equivalent to something in the image of $\underline{\mathbf{A}}$, since they are both a weak opposite of the same object. Hence $\underline{\mathbf{A}} \rightarrow \underline{\mathbf{A}}'$ is bicategorically essentially surjective, and thus a biequivalence. \square

Finally, we can prove [Theorem 2.3](#).

Theorem 8.3. *If \mathcal{A} is a bicategory with a weak duality involution, then there is a 2-category \mathcal{A}' with a strict duality involution and a duality pseudofunctor $\mathcal{A} \rightarrow \mathcal{A}'$ that is a biequivalence.*

Proof. By [Theorem 7.2](#), we can regard \mathcal{A} as a bicategory with contravariance $\underline{\mathbf{A}}$ having weak opposites. By [Theorem 8.1](#), it is therefore biequivalent to a 2-category with contravariance and weak opposites, and therefore by [Theorem 8.2](#) also biequivalent to a 2-category with contravariance and strict opposites.

Now by [Theorem 4.11](#) and [Theorem 5.2](#), the latter is equivalently a 2-category with a strong duality involution. Thus, by [Theorem 3.3](#) it is equivalent to a 2-category with a strict duality involution, say \mathcal{A}' . So we have a biequivalence $\mathcal{A} \rightarrow \mathcal{A}'$ that is a pseudofunctor preserving contravariance, and by [Theorem 7.4](#), we can also regard it as a duality pseudofunctor. \square

As mentioned in [section 1](#), we could actually dispense with the right-hand side of the ladder as follows. Let \mathcal{A}' be the full sub- \mathbf{V} -bicategory of \mathcal{MA} , as in [Theorem 8.1](#), consisting of the modules that are *either* of the form Y_z or of the form Y_z° , where Y_z° is defined by $(Y_z^\circ)^+(x) := \underline{\mathbf{A}}^-(x, z)^{\text{op}}$ and $(Y_z^\circ)^-(x) := \underline{\mathbf{A}}^+(x, z)^{\text{op}}$. This \mathcal{A}' is a 2-category with a strict duality involution that interchanges Y_z and Y_z° , and the Yoneda embedding is a biequivalence for the same reasons. However, this quicker argument still depends on the description of weak duality involutions using bicategorical enrichment from [section 7](#), and thus still depends *conceptually* on the entire picture.

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